Recurrence Relations

A **recurrence relation** for a sequence

\[ x_0, x_1, x_2, \ldots \]

is an expression for \( x_n \) in terms of \( x_0, x_1, \ldots, x_{n-1} \).

Very often we will express \( x_n \) in terms of the previous two or three values \( x_{n-1}, x_{N-2}, \) etc. but for the Catalan numbers there is a recurrence relation which uses all previous values:

\[ c_{n+1} = c_0c_n + c_1c_{n-1} + \cdots + c_nc_0. \]

This will be proved later. However, if we know suitable starting values (in this case \( c_0 = 1 \)) we can use the recurrence relation to calculate the sequence to as many terms as we please. For the Catalan numbers we find:

\[ 1, 1, 2, 5, 14, 42, \ldots \]
Recurrence Relations, Generating Functions and Closed Forms

Let \( L_n \) be the maximum number of regions in the plane produced by \( n \) lines. Then the \( L_n \) satisfy the recurrence relation

\[
L_n = L_{n-1} + n \quad \text{for } n \geq 1,
\]

with \( L_0 = 1 \).

The generating function for the sequence \( L_0, L_1, L_2, \ldots \) is

\[
L(z) = L_0 + L_1 z + L_2 z^2 + L_3 z^3 + \cdots.
\]

After multiplying this by \( z \) we find that

\[
zL(z) = L_0 z + L_1 z^2 + L_2 z^3 + \cdots.
\]

Now we can subtract \( zL(z) \) from \( L(z) \) to obtain

\[
L(z) - zL(z) = L_0 + (L_1 - L_0)z + (L_2 - L_1)z^2 + (L_3 - L_2)z^3 + \cdots.
\]
But we know that $L_0 = 1$ and that $L_n - L_{n-1} = n$, therefore

$$(1 - z)L(z) = 1 + z + 2z^2 + 3z^3 + \cdots = 1 + z(1 + 2z + 3z^2 + \cdots).$$

Using a the formula for $(1 - z)^{-2}$ from last lecture we see that

$$(1 - z)L(z) = 1 + z(1 - z)^{-2}.$$ 

Therefore a closed form for $L(z)$ is

$$L(z) = (1 - z)^{-1} + z(1 - z)^{-3}.$$ 

The formulas for $(1 - z)^{-1}$ and $(1 - z)^{-3}$ allow us to expand the closed form of $L(z)$ that we have just found. Thus

$$L(z) = \sum_{n=0}^{\infty} z^n + z \left( \sum_{m=0}^{\infty} \binom{m+2}{2} z^m \right)$$

$$= \sum_{n=0}^{\infty} z^n + \sum_{m=0}^{\infty} \binom{m+2}{2} z^{m+1}$$

$$= \sum_{n=0}^{\infty} z^n + \sum_{n=1}^{\infty} \binom{n+1}{2} z^n,$$

where we have used the substitution $n = m + 1$ in the second summation of the last line. By definition, the coefficient of $z^n$ is $L_n$ and therefore $L_n = 1 + \binom{n+1}{2}$. 
Partial Fraction Expansions

It turns out that a large class of recurrence relations lead to generating functions that have a closed form which is a rational function; i.e., one polynomial divided by another.

The theory of partial fractions allows us to write any rational function as a sum of a polynomial and simple rational functions of the form

$$
\frac{A}{(1 - \lambda z)^k}.
$$

These simple rational functions can be expanded using the formula

$$(1 - \lambda z)^{-k} = \sum_{m=0}^{\infty} \binom{m + k - 1}{m} \lambda^m z^m.
$$

After adding these expansions, the coefficient of $z^n$ gives the solution to the original recurrence relation.