Geometric Mechanics: Assignment 1

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due Tuesday May 7, 4pm

1. Consider the following Hamiltonians (assume that the real constant $\alpha, \omega, \omega_1, \omega_2$ are non-zero):

   (I) $H = \frac{1}{2}\omega_1(p_1^2 + q_1^2) + \frac{1}{2}\omega_2(p_2^2 + q_2^2)$

   (II) $H = \frac{1}{2}\alpha(p_1^2 - q_1^2) + \frac{1}{2}\omega(p_2^2 + q_2^2)$

   (III) $H = \omega(p_2 q_1 - p_1 q_2) + \alpha(p_1 q_1 + p_2 q_2)$

   In each case: a) find the flow $\phi_H^t(x)$ of the Hamiltonian vector field $J\nabla H$, b) verify that $\phi_H^t(x)$ is a symplectic map for any fixed t, c) show that the time derivative of the flow evaluated at $t = 0$ is the Hamiltonian vector field $J\nabla H$, and d) determine the linear stability of the equilibrium at the origin.

2. For the Hamiltonians in 1) find all periodic orbits and their minimal period τ. Use the flow map to compute the period map $\phi_H^\tau(x)$ and find the eigenvalues of the Jacobian of this symplectic matrix evaluated at some point x_0 on each periodic orbit. Thus determine the linear stability of the periodic orbits. For (II) show that the eigenvalues of the Jacobian of the flow map are independent of the point x_0 on the periodic orbit.

3. Assuming that $\alpha > 0$ in example (III) find all initial conditions x_0 whose limit is the origin when $t \to +\infty$, i.e. $\lim_{t\to\infty} \phi_H^t(x_0) = 0$. Similarly find all initial conditions whose limit is the origin for $t \to -\infty$. Show that in both cases these initial conditions form a two-dimensional plane. Assuming in addition that $\omega > 0$, describe the shape and direction of orbits in these planes in the two cases as t increases from $t = 0$.

4. Verify that the flow of the Hamiltonian vector field with Hamiltonian $K = p_2 q_1 - p_1 q_2$ is 2π periodic. Write the flow in complex notation with appropriately chosen complex variables. Find the basic polynomial invariants of this flow and the relation between them. Compute the Poisson brackets between these invariants, and express them in terms of the invariants. Thus find the Poisson structure matrix J_P for reduction by this symmetry. Find its Casimir(s).

5. Use the invariants and Poisson structure from the previous exercise to reduce the Hamiltonian $H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\alpha(q_1^2 + q_2^2) + \beta(q_1^2 + q_2^2)(q_1 p_1 + q_2 p_2)$ by the S^1 symmetry $\phi_K^\theta(x)$ with K given in the previous question. Find all equilibria of the reduced system and find the eigenvalues of their linearisation. (This Hamiltonian can be interpreted as a model for a charged particle in a radially symmetric electric potential and an apparent vector potential.)

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