Geometric Mechanics: Problem Sheet 1

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1. Find the "standard" symplectic matrix J corresponding to the ordering

   (a) $x = (p_1, p_2, q_1, q_2)$

   (b) $x = (q_1, q_2, p_1, p_2)$

   (c) $x = (p_1, q_1, p_2, q_2)$

   such that $\dot{x} = J\nabla H$ are Hamilton's equations.
   In each case verify that $J^t = J^{-1} = -J$.
   Generalise to $2n$ dimensions.

   Compute the first 4 powers of the standard J_2 and use this to compute $e^{J_2 t}$.
   Then repeat the computation by diagonalising J_2.

2. Verify the flow properties explicitly for the flow of the 1:1 resonant harmonic oscillator. Use matrix block form whenever possible. What changes if you consider the general harmonic oscillator with Hamiltonian

$$H = \frac{\omega_1}{2}(p_1^2 + q_1^2) + \frac{\omega_2}{2}(p_2^2 + q_2^2)\,?$$

3. Finish the computation that shows that any two distinct orbits of the 1:1 resonance Harmonic Oscillator are linked in the energy surface S^3 by computing the distance $d^2 = \xi^2 + \eta^2$ of the intersection of the second orbit in stereographic projection $\Pi(\phi_H^t(x_0))$ with the plane $\zeta = 0$. Express d in terms of a single invariant and show that there is exactly one intersection with the interior of the unit disk. (Hint: Use the fact that we are on the energy surface $2h = 1 = \rho_1 + \rho_2$ for simplification).

4. Find the flow $\phi_H^t(x)$ for the Hamiltonian

$$H = \alpha p_1 q_1 + \beta p_2 q_2$$

   Verify that $\phi_H^t(x)$ is a symplectic map for any fixed t, and that the time derivative of the flow evaluated at $t = 0$ is the Hamiltonian vector field $J\nabla H$.

5. Find the energy surfaces $\mathcal{E}_h$ for $H = \frac{1}{2}(p_1^2 - q_1^2) + \frac{1}{2}(p_2^2 - q_2^2)$. Show that the orbits of this Hamiltonian are contained in 2-dimensional planes through the origin. Find the basic polynomial invariants of this flow and the relation between them. Thus find the analogue of the Hopf map in this setting.

   Warning: the energy surfaces in this and the previous example are not compact!