Geometric Mechanics: Problem Sheet 3

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1. (Poincaré map) Consider the Hamiltonian $H = \frac{1}{2}(p_1^2 + 3q_1^2) + \frac{1}{2}(p_2^2 - q_2^2)$. Find the Poincaré return map for the Poincaré section $q_1 = 0$ with $\dot{q}_1 > 0$ in the energy surface $\mathcal{E}_h$ (Procedure: Choose an initial condition (q_2, p_2) on the section, find p_1 such that the initial condition x_0 is on $\mathcal{E}_h$. Then find the return time T such that $\phi_H^T(x_0)$ is again in the section with $\dot{q}_1 > 0$. This will give a linear map of the (q_2, p_2) plane to itself). Diagonalise this map by a linear symplectic transformation of the (q_2, p_2) plane. Thus find a conserved quantity of this map and express it in the original coordinates.

2. (general coordinate transformation) Consider the change of variables given by $(y_1, y_2) = \phi(x_1, x_2) = (x_2 + x_1^2 + a, -bx_1)$. Invert the transformation. Transform the ODE $\dot{x}_1 = cx_2, \dot{x}_2 = dx_1$ with this transformation. When $c = -d$ this ODE is Hamiltonian with a quadratic Hamiltonian and the standard J_2. Find $H(x)$, then transform the ODE as a Hamiltonian system, hence find $\tilde{J}(y) = D\phi(x) J_2 D\phi^t(x)$ and verify $\dot{y} = \tilde{J}\nabla_y H(y)$. What is the condition for $\tilde{J} = J$, i.e. that the transformation is symplectic?

3. (symplectic transformations) Show that the matrices

$$A_1 = \begin{pmatrix} M & 0 \\ 0 & M^{-t} \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} I & 0 \\ S & I \end{pmatrix}, \quad S^t = S$$

are symplectic such that the product $A_1 A_2 = \begin{pmatrix} M & 0 \\ M^{-t}S & M^{-t} \end{pmatrix}$ is also symplectic. Such symplectic matrices appear as Jacobians of so called *point transformations* from old variables (q, p) to new variables (Q, P) given by $Q = f(q)$, $P = (Df(q))^{-t}p$. Consider a point transformation with $f(q_1, q_2) = (\sqrt{q_1^2 + q_2^2}, \arctan(q_2/q_1))$ and show that the Jacobian $\partial(Q, P)/\partial(q, p)$ is of the form $A_1 A_2$ as above, and hence is symplectic.

4. (Poisson structure preserving coordinate transformations) From the lecture we know that the flow ϕ_H of the ODE $\dot{x} = J_P(x)\nabla H(x)$ (where J_P is a Poisson structure matrix) is a Poisson structure preserving map, i.e.

$$D\phi_H(x) J_P(x) D^t\phi_H(x) = J_P(\phi_H(x)) \,.$$

Consider the particular example $J_E(x)$ from the lectures together with a linear Hamiltonian $H = (v, x) = v_1x_1 + v_2x_2 + v_3x_3$. Show that the flow ϕ_H^t of this Hamiltonian (with Poisson structure $J_E(x)$) is a rotation about the axis v by the amount $|vt|$. Since v is arbitrary this shows that arbitrary rotations are Poisson structure preserving maps for $J_E(x)$. Show that the Poisson structure preserving property in this particular case is equivalent to

$$R(x \times R^t y) = (Rx) \times y$$

for arbitrary vectors $x, y \in \mathbf{R}^3$ and an arbitrary orthogonal matrix R. [1]

5. (Poisson map between canonical and non-canonical brackets) Use formulae (J.16), (J.8), (J.15) in the MATH3977 lecture notes to define a mapping ϕ from the old canonical variables $(\phi, \theta, \psi, p_\phi, p_\theta, p_\psi)$ to new non-canonical variables (J_1, J_2, J_3), $J_i = \mathbf{J}.\mathbf{e}_i$ (not to be confused with our symplectic matrix J). Find the Poisson structure matrix $\tilde{J}_P = D\phi\, J\, (D\phi)^t$ in the new variables. Equivalently, compute the three Poisson brackets $\{J_i, J_j\}$.

[1] Note that it is not true that $Ax \times Ay = A(x \times y)$, unless $A \in SO(3)$.