Geometric Mechanics: Problem Sheet 4

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1. Show that the map $\psi : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $(L_1, L_2, L_3) = (\sqrt{l^2 - p^2}\cos q, \sqrt{l^2 - p^2}\sin q, p)$ maps the Poisson structure $J_E = L\times$ into the standard symplectic structure, thus giving canonical coordinates on the sphere S^2 of radius l^2 (with two points removed) defined by fixing the Casimir $L_1^2 + L_2^2 + L_3^2$.

2. (a) Show that $\phi^s(x_1, x_2) \mapsto (e^s x_1, e^{\nu s} x_2)$, $\nu \in \mathbb{R}$ is a a group action of the (additive) group $\mathbb{R}$ on $\mathbb{R}^2$ with coordinates x_1, x_2, i.e. ϕ^s is a symmetry. (b) Find the ODE for which ϕ^s is the flow and determine for which ν this is a Hamiltonian flow and find the Hamiltonian. (c) Show that for $\nu = -2$ the above symmetry ϕ^s considered as a coordinate transformation leaves the ODE $\dot x_1 = x_1^3 x_2$, $\dot x_2 = a(x_1 x_2)^2$ invariant. (d) Use invariants to reduce this ODE (this is not a Hamiltonian ODE, so there is only reduction by one dimension).

3. Consider the motion of N points q_i, $i = 1, \ldots, N$, with masses m_i in $\mathbb{R}^3$ under mutual attraction derived from a potential V that depends on the distance between points only. The phase space is $\mathbb{R}^{6N}$ with coordinates $(q_1, \ldots, q_N, p_1, \ldots, p_N)$ where each q_i and p_i is a vector in $\mathbb{R}^3$. The symmetry group of this problem is the special Euclidean group $SE(3)$ generated by translations $\phi^\xi(q_1, \ldots, q_N, p_1, \ldots, p_N) = (q_1 + \xi, \ldots, q_N + \xi, p_1, \ldots, p_N)$ for $\xi \in \mathbb{R}^3$ and rotations $\phi^R(q_1, \ldots, q_N, p_1, \ldots, p_N) = (Rq_1, \ldots, Rq_N, Rp_1, \ldots, Rp_N)$ for $R \in SO(3)$. The Hamiltonian is $H = \frac{1}{2}\sum \frac{1}{m_i}|p_i|^2 + \sum_{i<j} V(|q_i - q_j|)$. Show that H is a Hamiltonian system with $SE(3)$ symmetry.

4. Consider a group G and denote two elements by g and h. (a) Show that the map $\phi^g(h) = gh$ (called left translation) defines a group action of G on itself. Show that this action is free (i.e. has no fixed points). (b) Show that the map $\phi^g(h) = ghg^{-1}$ (called conjugacy) defines a group action of G on itself. (Orbits of this group action are called conjugacy classes.) Show that for $G = GL(n)$ (the general linear group of invertible matrices) this action is not free.

5. (a) Show that $Ad_A(\xi) = A\xi A^{-1}$ for $A \in Sp(2n)$ and $\xi \in \mathfrak{sp}(2n)$ maps elements of the Lie algebra into elements of the Lie algebra using that A is symplectic and that $\xi = JS$ where $S^t = S$. (b) Show that $Ad_A(\xi)$ for the special linear group defined by $\det A = 1$ maps matrices ξ with vanishing trace into matrices with vanishing trace.

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