Symplectic group $Sp(2n)$

$GL(n)$: set of $n \times n$ matrices with $\det \neq 0$

Def: $Sp(2n) = \{A \in GL(2n) : A^t J A = J\}$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

Note: $J^2 = -I_{2n}$, $J^t = J^{-1} = -J$

Thm: $Sp(2n)$ is a group.

- $I \in Sp(2n)$? $\quad I^t J I = J$ ✓
- $A_1, A_2 \in Sp(2n) \overset{?}{\Longrightarrow} A_1 A_2 \in Sp(2n)$?

$$(A_1A_2)^t J (A_1A_2) = A_2^t \underbrace{A_1^t J A_1}_{} A_2$$
$$= A_2^t J A_2 = J$$

- $A \in Sp(2n) \overset{?}{\Longrightarrow} A^{-1} \in Sp(2n)$?

$$A^t J A = J \Rightarrow A^t J = J A^{-1}$$
$$\Rightarrow J^{-1} A^t J = A^{-1}$$

$$(J^{-1}A^tJ)^t J (J^{-1}A^tJ) = J^t \underbrace{A J A^t}_{=J} J \quad \to \text{exercise}$$
$$= J$$

✓

- associative ✓

Similarly: $SO(n) = \{R \in GL(n) : R^t R = I, \det R = 1\}$

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Creating Sp(2n)

Any quadratic Hamiltonian $H: \mathbb{R}^{2n} \to \mathbb{R}$

$x \in \mathbb{R}^{2n}$

$H = \frac{1}{2}(x, Sx)$, $S^t = S$, $n \times n$ matrix.

ODE: $\dot{x} = J\nabla H = JSx = Bx$ linear

flow: $\phi^t_H(x) = e^{JSt}x$

Theorem: $A(t) = e^{JSt} \in Sp(2n)$ for any symmetric S, any t.

$U(t) = A^t(t) J A(t)$

$U(0) = I^t J I = J$ ✓

Show $\dot{U} = 0$, then $U(t) = J$

$\dot{U} = \dot{A}^t J A + A^t J \dot{A}$, $\dot{A} = BA$

$= (BA)^t J A + A^t J B A$

$= A^t (B^t J + J B) A$

$= A^t (S^t J^t J + J J S) A$

$= A^t (S - S) A = 0$ ✓

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for SO(n): $\dot{x} = Bx$, $R = e^{Bt}$, $\dot{R} = BR$

$U(t) = R^t R$, find B s.t. $U(t) = I$

Sp(2n), SO(n) are Lie groups

Corresponding Lie algebras are

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$$sp(2n) = \{ B_{2n\times 2n} : B^t J + J B = 0 \}$$
$$= \{ B : B = JS,\ S^t = S \}$$

$$so(n) = \{ B : \quad \cdots\cdots \}$$

$$\dim Sp(2n) = \dim sp(2n)$$
$$= n(2n+1)$$

<u>Claim</u>: $A \in Sp(2n) \Rightarrow \det A = 1$

$$J = A^t J A$$
$$\det J = \det A^t \det J \det A \Rightarrow$$

$1 = (\det A)^2$, see e.g. Meyer & Hall

$n=1$: $Sp(2) = \{ A : \det A = 1 \}$

area preserving transformations

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symplectic form ω

$\omega(x,y) = (x, Jy)$, $\omega(x,y) = -\omega(y,x)$

linear symplectic map A preserves ω:

$$\omega(Ax, Ay) = (Ax, JAy) = (x, A^t JAy)$$
$$= (x, Jy) = \omega(x,y).$$