Recall: $Sp(2n) = \{A \in GL(2n) : A^t J A = J\}$ $\quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$

$H = \frac{1}{2}(x, Sx)$: $e^{JSt} \in Sp(2n)$, $S^t = S$

$B = JS \iff$

finish: $B^t J + JB = 0$

$sp(2n) = \{B_{2n\times 2n} : B^t J + JB = 0\}$

n=1: $Sp(2) = \{A_{2\times 2} : \det A = 1\}$ area preserving exp

$sp(2) = \{B_{2\times 2} : \operatorname{tr} B = 0\}$

Symplectic form: $\omega(x,y) = (x, Jy)$, $J^t = -J$, $\det J \neq 0$

$\omega(x,y) = -\omega(y,x)$

Linear stability of Ham. equilibrium pts.

$\dot{x} = J\nabla H$ equilibrium at x_0

$J\nabla H(x_0) = 0 \Rightarrow \nabla H(x_0) = 0$

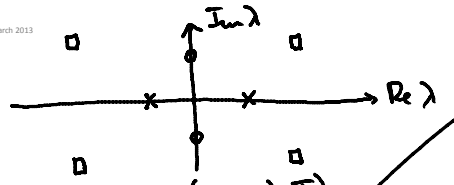
Taylor series at x_0 (Hessian)

$H(x_0 + \xi) = H(x_0) + (\nabla H(x_0), \xi) + \frac{1}{2}(\xi, D^2H(x_0)\xi) + \text{h.o.t.}$

$\frac{d}{dt}(x_0 + \xi) = J\nabla H(x_0 + \xi)$

$\dot{\xi} = JD^2H(x_0)\xi + \text{h.o.t.} = B\xi + \text{h.o.t.}$

Thm: If λ is an eigenvalue of $B = JS$, $S^t = S$ then so is $-\lambda$, $\bar{\lambda}$, $-\bar{\lambda}$ (S real)



$\det A^t = \det A$

$B^t J + J B = 0$

$J B^t J - B = 0$

$\det J = 1$

Proof: $P(\lambda) = \det(B - \lambda I)$

$= \det(B^t - \lambda I)$

$= \det(J(B^t - \lambda I)J)$

$= \det(JB^tJ + \lambda I)$

$= \det(B + \lambda I) = P(-\lambda)$ P is even

$\therefore P(\lambda) = 0 \Rightarrow P(-\lambda) = 0$

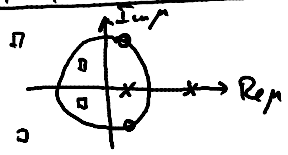
P is real $\Rightarrow P(\lambda) = 0 \Rightarrow P(\bar{\lambda}) = 0$ □

Conclusion: Equilibria of Ham. S. cannot be asymptotically stable (all $\mathrm{Re}(\lambda) < 0$). They are linearly stable if all $\mathrm{Re}(\lambda) = 0$ and B is diagonalisable.

Hamiltonian $B = JS$ $\xrightarrow{\exp.}$ Symplectic $A = e^{Bt}$

$\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda}$ $\quad \mu = e^{\lambda t}, \frac{1}{\mu} = e^{-\lambda t}$

$\bar{\mu} = e^{\bar{\lambda} t}, \frac{1}{\bar{\mu}} = e^{-\lambda t}$



possibly stable if all $|\mu| = 1$

Tuesday, 12 March 2013
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Example: Henon-Heiles system (star in galaxy)

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + \tfrac{1}{2}(q_1^2 + q_2^2) + q_1^2 q_2 - \tfrac{1}{3} q_2^3$$

$$\nabla H = (q_1 + 2q_1q_2,\ q_2 + q_1^2 - q_2^2,\ p_1, p_2)^t = 0$$

$p_1 = 0,\ p_2 = 0$ ← $q_1(1 + 2q_2) = 0$ → $q_2 = -\tfrac{1}{2}$

$q_1 = 0$ $\qquad$ $q_1^2 = \tfrac{3}{4}$

$q_2(1 - q_2) = 0$

$x_0 = (0, 0, 0, 0)$ $\qquad$ $x_{2,3} = (\pm\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}, 0, 0)$

$x_1 = (0, 1, 0, 0)$

$$D^2H = \begin{pmatrix} 1+2q_2 & 2q_1 & 0 & 0 \\ 2q_1 & 1-2q_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix}$$

$$B = JD^2H = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & I \\ -Q & 0 \end{pmatrix}$$

$q_1 = 0$ $\quad \det(B - \lambda I) = \left|\begin{pmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -a & 0 & -\lambda & 0 \\ 0 & -b & 0 & -\lambda \end{pmatrix}\right| = (\lambda^2 + a)(\lambda^2 + b)$

imaginary eigenvalues if $a \geq 0,\ b \geq 0$