Recall: $A \in Sp(2n,\mathbb{R})$, μ eigenvalue of A
then so are $\frac{1}{\mu}$, $\bar{\mu}$, $\frac{1}{\bar{\mu}}$.

Stability for periodic orbit through x_0, period τ:

$\phi_H^\tau(x_0) = x_0$, Jacobian $\frac{\partial \phi_H^\tau(x_0)}{\partial x_0} = A \in Sp(2n,\mathbb{R})$
is called Monodromy matrix
eigenvalues μ are called multipliers.

Example: $H = \frac{1}{2}(p_1^2 + 3q_1^2) + \frac{1}{2}(p_2^2 - q_2^2)$, $x = (q_1, p_1, q_2, p_2)$

$\dot{x} = J\nabla H = JSx$, $S = D^2H = \begin{pmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$ $J = \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix}$ $J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$A(t) = e^{JSt} = \begin{pmatrix} \cos\sqrt{3}t & \sqrt{3}\sin\sqrt{3}t & & \\ -\sqrt{3}\sin\sqrt{3}t & \cos\sqrt{3}t & & 0 \\ & & \cosh t & \sinh t \\ 0 & & \sinh t & \cosh t \end{pmatrix}$

$\phi_H^t(x) = A(t)x$, find x_0, τ s.t. $A(\tau)x_0 = x_0$

$\Rightarrow$ $x_0 = \begin{pmatrix} q_1 \\ p_1 \\ 0 \\ 0 \end{pmatrix}$, $\tau = \frac{2\pi}{\sqrt{3}}$

Monodromy matrix $A(\tau) = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & \cosh\tau & \sinh\tau \\ 0 & & \sinh\tau & \cosh\tau \end{pmatrix}$

multipliers $\mu = 1, 1, e^{\pm\tau}$; $e^\tau > 1 \Rightarrow$ unstable.

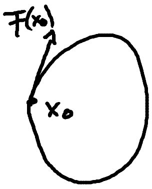
Theorem: $D\phi_H^\tau(x_0)$ has eigenvector $J\nabla H(x_0)$ and left eigenvector $\nabla H(x_0)$ with eigenvalue 1 (assuming $\phi_H^\tau(x_0) = x_0$)

Proof: $\dot{x} = F(x)$, $\phi^\tau(x_0) = x_0$ periodic

$$\frac{\partial}{\partial t}\left(\phi^\tau(\phi^t(x_0)) = \phi^{\tau+t}(x_0)\right)$$ flow property

$$\frac{\partial\phi^\tau}{\partial x}\frac{\partial\phi^t(x_0)}{\partial t} = \frac{\partial\phi^{\tau+t}(x_0)}{\partial t}$$ set $t=0$

$$\left.\frac{\partial\phi^\tau}{\partial x}\right|_{x=x_0} F(x_0) = F(x_0)$$



so $F(x_0)$ is eigenvector with $\mu=1$

In Ham. syst. $F(x_0) = J\nabla H(x_0)$.

displacement along orbit has no effect

Integral $G(x)$ = const. along solutions

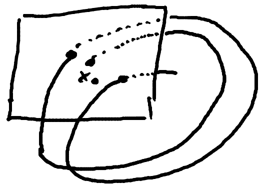
$$\frac{\partial}{\partial x_0}\left(G(\phi^t(x_0)) = G(x_0)\right), \quad t=\tau$$

$$\frac{\partial G}{\partial x_0}\frac{\partial\phi^\tau}{\partial x_0} = \frac{\partial G}{\partial x_0}$$ ↖ row vector

H is integral (if autonomous) of ϕ_H^t, so ∇H is left eigenvector with eigenvalue 1.

Poincare section :

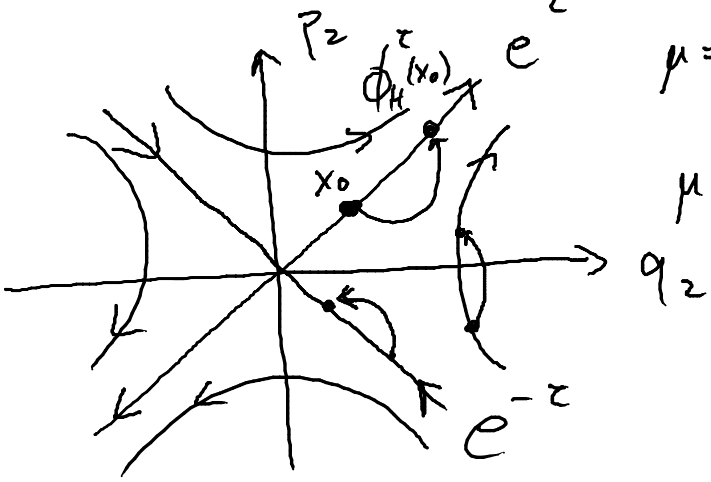
& fix energy



Example: Poincaré section $q_1 = 0$

$\Sigma_h = \{ x \in \mathbb{R}^4 : H(x) = h \}$

solve for p_1



$\mu = e^{\tau}$ eigenvector $\left(\begin{array}{c} 0 \\ 0 \\ \hline 1 \\ 1 \end{array}\right)$

$\mu = e^{-\tau}$ eigenvector $\left(\begin{array}{c} 0 \\ 0 \\ \hline -1 \\ 1 \end{array}\right)$

restrict to q_2, p_2

2DOF hyperbolic periodic orbit