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Hamiltonian Noether Theorem

ϕ^s one parameter symmetry of H: $H \circ \phi^s = H$

If $\phi^s = \phi^s_K$ then K is an integral of H:

$\dot{K} = \{K, H\} = 0$, or $K \circ \phi^t_H = K$.

$$\frac{d}{ds} H \circ \phi^s_K = \{H, K\} = 0 \quad \text{(last week)}$$

$$\Rightarrow \{K, H\} = 0 \qquad \square$$

K is constant w.r.t. the flow of H.

H is constant w.r.t. the flow of K.

Reduction:

1) fix integral $\mathcal{E}_k = \{x : K(x) = k\} = K^{-1}(k)$
2) identify points on the same orbit ϕ^s_K

1+2: reduction by 2 dimensions.

How to do that in practice?

Use invariants of ϕ^s_K as coordinates!

Recall week 1: $\mathbb{R}^4$, $z_j = q_j + i p_j$, $j = 1, 2$

$K = \frac{1}{2}|z_1|^2 + \frac{1}{2}|z_2|^2$, $\phi^s_K(z_1, z_2) = (e^{is} z_1, e^{is} z_2)$

where $H = f(s_1, s_2, s_3, s_4)$ s.t. $H \circ \phi^s_K = H$

$s_1 = |z_1|^2$, $s_2 = |z_2|^2$, $s_3 = \mathrm{Re}(z_1 \bar{z}_2)$, $s_4 = \mathrm{Im}(z_1 \bar{z}_2)$

Reduce to $\Sigma_k = \{x : K(k) = k\} = S^3$

quotient $M_k = \Sigma_k / S^1_K$ identify pts on the same ϕ^s_K orbit

$$\boxed{\begin{aligned} w_1 &= q_1 q_2 + p_1 p_2 \\ w_2 &= -q_1 p_2 + q_2 p_1 \end{aligned}}$$

Hopf map $\tilde{\mathcal{H}} : \Sigma_k \to S^2$

$w_1 = S_3$, $w_2 = S_4$, $w_3 = \frac{1}{2}(S_2 - S_1)$, $w_4 = \frac{1}{2}(S_2 + S_1) = k$

$$\{w_1, w_2\} = \frac{\partial w_1}{\partial q_1}\frac{\partial w_2}{\partial p_1} - \frac{\partial w_1}{\partial p_1}\frac{\partial w_2}{\partial q_1} + \frac{\partial w_1}{\partial q_2}\frac{\partial w_2}{\partial p_2} - \frac{\partial w_1}{\partial p_2}\frac{\partial w_2}{\partial q_2}$$

$$= q_2^2 - p_2(-p_2) + q_1(-q_1) - p_1^2$$

$$= q_2^2 + p_2^2 - q_1^2 - p_1^2 = (S_2 - S_1) = 2w_3$$

etc. ... $\{w_4, w_i\} = 0$

$$J_p = \begin{pmatrix} 0 & 2w_3 & -2w_2 \\ -2w_3 & 0 & 2w_1 \\ 2w_2 & -2w_1 & 0 \end{pmatrix}$$ Poisson structure on $\mathbb{R}^3$ with coords w_1, w_2, w_3

J_p has Casimir $C = w_1^2 + w_2^2 + w_3^2$

$J_p \nabla C = 0$, fix Casimir $\{C = c\} = S^2$

$\tilde{\mathcal{H}}$ is a Poisson map

reduced dynamics : $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

$$\dot{w} = J_p \nabla_w H$$ Hamiltonian dynamical system on S^2

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Remarks on Reduction

- invariants are usually related
- the may satisfy inequalities

$$s_i = |z_i|^2 \geq 0$$

- reduced Hamiltonian is the original Hamiltonian written in invariants.

Example: Free rigid body (Euler top)

example with symmetry group $SO(3)$

Convention: X in body, x in space (in $\mathbb{R}^3$)

$$x = R X, \quad R \in SO(3), \quad RR^T = I$$

$$\dot{x} = \dot{R} X \qquad X = \text{const.}$$

$$= \dot{R} R^T x = \hat{\omega} x = \omega \times x$$

antisymmetric matrix

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}, \quad \hat{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

$$\omega = R \Omega$$

claim: $\hat{\Omega} = R^t \dot{R}$

Identity: $\widehat{Ra} = R \hat{a} R^t$

$$\left.\right\} \hat{\Omega} = \widehat{R^t \omega} = R^t \hat{\omega} R = R^t \dot{R} R^t R = R^t \dot{R} \quad \checkmark$$

$$\widehat{Ra}\, b = (Ra) \times b = R(a \times R^t b) = R \hat{a} R^t b \quad \checkmark$$

configuration space $SO(3) \ni R$

$\phi^g(R, \dot{R}) = (gR, g\dot{R})$, $g \in SO(3)$

observe: $\hat{\Omega} \circ \phi^g = \hat{\Omega}$ invariant

$(gR)^+ g\dot{R} = R^+ g^+ g \dot{R} = R^+ \dot{R}$ ✓

Tensor of inertia Θ, $\Theta^+ = \Theta$

$L = \Theta \Omega$ angular momentum, invariant under ϕ^g

equations of motion: $L = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix}$ $\quad J_E = \hat{L}$

$\dot{L} = L \times \Omega = L \times \Theta^{-1} L = J_E \nabla_L H$, $H = \frac{1}{2}(L, \Theta^{-1} L)$

(kinetic energy)

finding R: $\hat{\Omega} = R^+ \dot{R} \Rightarrow \dot{R} = R\, \widehat{\Theta L}$

Θ diagonal $= \mathrm{diag}(I_1, I_2, I_3)$

$I_1 > I_2 > I_3$

$$\begin{pmatrix} \dot{L}_1 \\ \dot{L}_2 \\ \dot{L}_3 \end{pmatrix} = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} \times \begin{pmatrix} L_1/I_1 \\ L_2/I_2 \\ L_3/I_3 \end{pmatrix} = \begin{pmatrix} L_2 L_3 \left(\frac{1}{I_3} - \frac{1}{I_2}\right) \\ L_1 L_3 \left(\frac{1}{I_1} - \frac{1}{I_3}\right) \\ L_1 L_2 \left(\quad \right) \end{pmatrix} = \begin{pmatrix} \alpha L_2 L_3 \\ -\beta L_1 L_3 \\ \gamma L_1 L_2 \end{pmatrix}$$

$\alpha, \beta, \gamma \geq 0$

Equilibria:

$L_1 = L_2 = 0$: $L = \mu \underset{\sim}{k}$

$L_1 = L_3 = 0$: $L = \mu \underset{\sim}{j}$

$L_2 = L_3 = 0$: $L = \mu \underset{\sim}{i}$

lin. stability

$$DJ_E \nabla H = \begin{pmatrix} 0 & \alpha L_3 & \alpha L_2 \\ -\beta L_3 & 0 & -\beta L_1 \\ \gamma L_2 & \gamma L_1 & 0 \end{pmatrix}$$

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$L = \mu \underset{\sim}{i}$

$$D J_E \nabla H(\mu \underset{\sim}{i}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mu\beta \\ 0 & \mu\gamma & 0 \end{pmatrix}$$

char. poly. $\lambda(\lambda^2 + \mu^2 \beta\gamma) \Rightarrow$ lin. stable

$L = \mu \underset{\sim}{j}$ ... $\lambda(\lambda^2 - \mu^2 \alpha\gamma) \Rightarrow$ unstable

$L = \mu \underset{\sim}{k}$ $\lambda(\lambda^2 + \mu^2 \alpha\beta) \Rightarrow$ lin. stable

Casimir $L_1^2 + L_2^2 + L_3^2 = \mu^2$ sphere

Hamiltonian $\frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3} = 2h$ ellipsoid