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Lie algebras

Lie group G : group operation and inverse are smooth
Restrict to matrix groups

Ex: $Sp(2n)$ $\quad O(n) \quad Sl(n) \quad U(n)$

$A^t J A = J \quad AA^t = I \quad \det A = 1 \quad A\bar{A}^t = I$

Let $A(s)$ be a path of matrices in G with $A(0) = I$:

$$A(s) = I + sA'(0) + \frac{s^2}{2}A''(0) + \cdots \quad \text{Taylor series}$$

$$\frac{d}{ds}A(s)\Big|_{s=0} = A'(0)\ .$$ All such $A'(0)$ form the Lie algebra $\mathfrak{g}$. Notation $\xi, \eta \in \mathfrak{g}$

(previously: B, $A = e^{Bs}$, where $B = JS$, $S^t = S$)

We say path $e^{\xi s} = I + s\xi + \frac{s^2}{2}\xi^2 + \cdots$ is generated by ξ.

$$\frac{d}{ds}e^{\xi s}\Big|_{s=0} = \xi\ , \quad \frac{d}{ds}e^{\xi s} = \xi e^{\xi s}$$

(cmp. with flow properties)

Claim: If $\xi, \eta \in \mathfrak{g}$, then $[\xi, \eta] = \xi\eta - \eta\xi \in \mathfrak{g}$

[Ex: $SO(n)$: $\xi^t = -\xi$, $\eta^t = -\eta$, $[\xi,\eta]^t = (\xi\eta - \eta\xi)^t = \eta^t\xi^t - \xi^t\eta^t$
$= -[\xi,\eta]$ ✓]

conjugation action: $gAg^{-1} \in G$, $A, g \in G$

so $$\frac{d}{ds}gA(s)g^{-1}\Big|_{s=0} = gA'(0)g^{-1} \in \mathfrak{g}$$

adjoint action $Ad_g\xi = g\xi g^{-1} \in \mathfrak{g}$ for $g \in G$, $\xi \in \mathfrak{g}$

Consider $g = e^{\eta t}$, $\eta \in \mathfrak{g}$, then

$$e^{\eta t}\xi e^{-\eta t} \in \mathfrak{g} \qquad \text{expand in } t$$

$$(1 + \eta t + \cdots)\xi(1 - \eta t + \cdots) = \xi + t(\eta\xi - \xi\eta) + O(t^2)$$

so $$\frac{d}{dt}Ad_{e^{\eta t}}\xi\Big|_{t=0} = [\eta, \xi] \in \mathfrak{g}$$

Lie algebra is closed under Lie bracket

$[\,,\,]$ which is
1) bilinear
2) anti-symmetric
3) Jacobi Identity (from assoc. of Matrix mult.)

Notation: lower case for algebra $so(n)$

Ex: 1) $so(n) = \{ \xi \in Mat(n) : \xi^t = -\xi \}$

2) $sp(2n) = \{ \xi : \xi = JS, \; S^t = S \}$

$\eta = JT, \quad T^t = T$

$[\xi,\eta] = [JS, JT] = JSJT - JTJS$

$= J(\underbrace{SJT - TJS}_{\text{symmetric!}})$

$(SJT - TJS)^t = T^t J^t S^t - S^t J^t T^t$

$= T(-J)S - S(-J)T = SJT - TJS \checkmark$

Ad for $sp(2n)$ $\quad \xi = JS, \; S^t = S, \; A \in Sp(2n)$

$Ad_A \xi = A \xi A^{-1} = AJSA^{-1} = -JJAJSA^{-1}$ with $\underbrace{JAJSA^{-1}}_{\text{symmetric!}}$

$(JAJSA^{-1})^t = A^{-t} S J A^t J$, use $A^t J A = J$

$= + JAJS A^{-1} \checkmark$

3) $so(3) \quad \xi^t = -\xi \qquad [\,,\,] \qquad Ad_R \xi = R \xi R^t$

$\wedge \uparrow ①$ $\qquad \uparrow ②$

$so(3) \quad a \in \mathbb{R}^3 \qquad \times \qquad \widetilde{Ad}_R a = Ra$

① $\hat{a} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

② $[\hat{a}, \hat{b}] = \widehat{a \times b}$

③ $Ad_R \hat{a} = R \hat{a} R^t = \widehat{\widetilde{Ad}_R a} = \widehat{Ra}$

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(4) $$[Ad_R \hat{a}, Ad_R \hat{b}] = Ad_R [\hat{a}, \hat{b}]$$

$$\Downarrow (2) \qquad\qquad \Downarrow (3)$$

$$(Ra) \times (Rb) = R(a \times b)$$