Factorisation algebras associated to Hilbert schemes of points

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Motivation

- **Learn about factorisation:**
  Provide and study examples of factorisation spaces and algebras of arbitrary dimensions.

- **Learn about Hilbert schemes:**
  Factorisation structures formalise the intuition that a space is built out of local bits in a specific way. Factorisation structures are expected to arise, based on the work of Grojnowski and Nakajima.
Outline

1. Main constructions: $\mathcal{H}ilb_{\text{Ran}X}$ and $\mathcal{H}_{\text{Ran}X}$

2. Chiral algebras

3. Results on $\mathcal{H}_{\text{Ran}X}$
Section 1

Main constructions: $\mathcal{H}ilb_{\text{Ran}X}$ and $\mathcal{H}_{\text{Ran}X}$
• Fix $k$ an algebraically closed field of characteristic 0.
• Let $X$ be a smooth variety over $k$ of dimension $d$.
• We work in the category of prestacks:

$$\text{PreStk} \coloneqq \text{Fun}(\text{Sch}^{\text{op}}, \infty\text{-Grpd})$$

$\xymatrix{\text{Sch} \ar@{^{(}<>(0.7)}[r] & \text{PreStk}}$

(Yoneda embedding)
Fix $n \geq 0$. The Hilbert scheme of $n$ points in $X$ is (the scheme representing) the functor

$$
\text{Hilb}_X^n : \text{Sch}^{\text{op}} \to \text{Set} \subset \infty\text{-Grpd}
$$

$$
S \mapsto \text{Hilb}_X^n(S),
$$

where

$$
\text{Hilb}_X^n(S) := \left\{ \xi \subset S \times X, \text{ a closed subscheme, flat over } S \right. \text{ with zero-dimensional fibres of length } n \right\}.
$$
The Hilbert scheme of points

Example: $k$-points

$$\text{Hilb}_X^n(\text{Spec } k) = \left\{ \xi \subset X \text{ closed zero-dimensional subscheme of length } n \right\}.$$ 

For example, for $X = \mathbb{A}^2 = \text{Spec } k[x, y]$, $n = 2$, some $k$-points are

$$\xi_1 = \text{Spec } k[x, y]/(x, y^2)$$
$$\xi_2 = \text{Spec } k[x, y]/(x^2, y)$$
$$\xi_3 = \text{Spec } k[x, y]/(x, y(y - 1)).$$

Notation: let $\text{Hilb}_X := \bigcup_{n \geq 0} \text{Hilb}_X^n$. 
The Ran space is a different way of parametrising sets of points in $X$:

$$\text{Ran } X(S) := \{ A \subset \text{Hom}(S, X), \text{ a finite, non-empty set } \} .$$

Let $A = \{ x_1, \ldots, x_d | x_i : S \rightarrow X \}$ be an $S$-point of Ran $X$.

For each $x_i$, let $\Gamma_{x_i} = \{ (s, x_i(s)) \in S \times X \}$ be its graph, and define

$$\Gamma_A := \bigcup_{i=1}^{d} \Gamma_{x_i} \subset S \times X,$$

a closed subscheme with the reduced scheme structure.
The Ran space

The Ran space is not representable by a scheme, but it is a pseudo-indscheme:

$$\text{Ran } X = \colim_{I \in \text{fSet}^{\text{op}}} X^I.$$ 

Here the colimit is taken in PreStk, over the closed diagonal embeddings

$$\Delta(\alpha) : X^J \hookrightarrow X^I$$

induced by surjections of finite sets

$$\alpha : I \to J.$$
Main definition: $\mathcal{H}ilb_{\text{Ran} \, X}$

Define the prestack $\mathcal{H}ilb_{\text{Ran} \, X} : \text{Sch}^{\text{op}} \to \text{Set} \subset \infty\text{-Grpd}$

$$S \mapsto \mathcal{H}ilb_{\text{Ran} \, X}(S)$$

by setting $\mathcal{H}ilb_{\text{Ran} \, X}(S)$ to be the set

$$\{(A, \xi) \in (\text{Ran} \, X \times \text{Hilb}_X)(S) \mid \text{Supp}(\xi) \subset \Gamma_A \subset S \times X\}.$$  

**Note:** This is a set-theoretic condition.

**Notation:** We have natural projection maps

$$f : \mathcal{H}ilb_{\text{Ran} \, X} \to \text{Ran} \, X,$$

$$\rho : \mathcal{H}ilb_{\text{Ran} \, X} \to \text{Hilb}_X.$$
$\mathcal{Hilb}_{\text{Ran} \, X}$ as a pseudo-indscheme

For a finite set $I$, we define

$$\mathcal{Hilb}_{X^I} : \text{Sch}^{\text{op}} \to \text{Grpd}$$

by setting $\mathcal{Hilb}_{X^I}(S) \subset (X^I \times \text{Hilb}_X)(S)$ to be

$$\left\{ ( (x_i)_{i \in I}, \xi) \mid (\{x_i\}_{i \in I}, \xi) \in \mathcal{Hilb}_{\text{Ran} \, X}(S) \right\}.$$

For $\alpha : I \to J$, we have natural maps

$$\mathcal{Hilb}_{X^J} \to \mathcal{Hilb}_{X^I},$$

defined by $((x_j)_{j \in J}, \xi) \mapsto (\Delta(\alpha)(x_j), \xi)$.

Then $\mathcal{Hilb}_{\text{Ran} \, X} = \text{colim}_{I \in \text{fSet}^{\text{op}}} \mathcal{Hilb}_{X^I}$. 
Factorisation

Consider \((\mathcal{H}ilb_{\text{Ran}X})_{\text{disj}} = \{(A = A_1 \sqcup A_2, \xi) \in \mathcal{H}ilb_{\text{Ran}X}\}\).

Suppose that in fact \(\Gamma_{A_1} \cap \Gamma_{A_2} = \emptyset\), so that if we set \(\xi_i := \xi \cap \widehat{\Gamma}_{A_i}\), we see that

1. \(\xi = \xi_1 \sqcup \xi_2\)
2. \((A_i, \xi_i) \in \mathcal{H}ilb_{\text{Ran}X}\) for \(i = 1, 2\).

Proposition

\[(\mathcal{H}ilb_{\text{Ran}X})_{\text{disj}} \simeq (\mathcal{H}ilb_{\text{Ran}X} \times \mathcal{H}ilb_{\text{Ran}X})_{\text{disj}}.\]
In particular, when \( A = \{x_1\} \sqcup \{x_2\} \), we can express this formally as follows:

- Set \( U := X^2 \setminus \Delta(X) \xrightarrow{j} X^2 \).
- Then the proposition specialises to the statement that there exists a canonical isomorphism

\[
c : \mathcal{Hilb}_{X^2} \times X^2 U \xrightarrow{\sim} (\mathcal{Hilb}_X \times \mathcal{Hilb}_X) \times X \times X U.
\]

We have similar isomorphisms \( c(\alpha) \) associated to any surjection of finite sets \( I \rightarrow J \). These are called factorisation isomorphisms.
Factorisation

**Theorem**

\[ f : \text{Hilb}_{\text{Ran}} X \rightarrow \text{Ran} X \text{ defines a factorisation space on } X. \text{ If } X \text{ is proper, } f \text{ is an ind-proper morphism.} \]
Linearisation of $\mathcal{Hilb}_{\text{Ran} \, X}$

**Set-up:** Let $\lambda^I \in \mathcal{D}(\mathcal{Hilb}_X)$ be a family of (complexes of) $\mathcal{D}$-modules compatible with the factorisation structure.

Then the family $\{\mathcal{A}_X^I := (f_i)_! \lambda^I \in \mathcal{D}(X^I)\}$ defines a factorisation algebra on $X$.

**More precisely:** For every $\alpha : I = \bigsqcup_{j \in J} I_j \to J$, we have isomorphisms

1. $\nu(\alpha) : \Delta(\alpha)^! \mathcal{A}_X^I \xrightarrow{\sim} \mathcal{A}_X^J$

   $\Rightarrow \{\mathcal{A}_X^I\}$ give an object “$\text{colim} \mathcal{A}_X^I$” of $\mathcal{D}(\text{Ran} \, X)$, which we’ll denote by $f_! \lambda$.

2. $c(\alpha) : j(\alpha)^* (\mathcal{A}_X^I) \xrightarrow{\sim} j(\alpha)^* (\bigsqcap_{j \in J} \mathcal{A}_X^{I_j})$
Linearisation of $\mathcal{Hilb}_{\text{Ran} \ X}$

**Definition**

Set $\mathcal{H}_{\mathcal{X} I} := (f_I)_! \omega_{\mathcal{Hilb}_{\mathcal{X} I}}$.

This gives a factorisation algebra

$$\mathcal{H}_{\text{Ran} \ X} = f_! \omega_{\mathcal{Hilb}_{\text{Ran} \ X}}.$$

Goal for the rest of the talk: study this factorisation algebra.
Section 2

Chiral algebras
A chiral algebra on $X$ is a $\mathcal{D}$-module $\mathcal{A}_X$ on $X$ equipped with a Lie bracket

$$\mu_\mathcal{A} : j_* j^* (\mathcal{A}_X \boxtimes \mathcal{A}_X) \to \Delta_! \mathcal{A}_X \in \mathcal{D}(X \times X).$$
Factorisation algebras and chiral algebras

**Theorem (Beilinson–Drinfeld, Francis–Gaitsgory)**

We have an equivalence of categories

\[
\left\{ \text{factorisation algebras} \right\}_{\text{on } X} \xrightarrow{\sim} \left\{ \text{chiral algebras} \right\}_{\text{on } X}.
\]
Idea of the proof

Let \( \{ \mathcal{A}_{X^I} \} \) be a factorisation algebra.

\[
\begin{array}{c}
\mathcal{A}_X \boxtimes \mathcal{A}_X \\
j_*j^* \rightarrow \\
\Delta! \Delta! \mathcal{A}_X^2
\end{array}
\]
Idea of the proof

Let \( \{A_X^I\} \) be a factorisation algebra.

\[
j_*j^* (A_X \boxtimes A_X) \to j_*j^* (A_X^2) \to \Delta! \Delta^! A_X^2 \to \Delta! \Delta^! A_X
\]

This defines \( \mu_A : j_*j^* (A_X \boxtimes A_X) \to \Delta! A_X \).

To check the Jacobi identity, we use the factorisation isomorphisms for \( I = \{1, 2, 3\} \).
Aside: chiral algebras and vertex algebras

Let \((V, Y(\cdot, z), |0\rangle)\) be a quasi-conformal vertex algebra, and let \(C\) be a smooth curve.

We can use this data to construct a chiral algebra \((\mathcal{V}_C, \mu)\) on \(C\).

This procedure works for any smooth curve \(C\), and gives a compatible family of chiral algebras. Together, all of these chiral algebras form a universal chiral algebra of dimension 1.
A Lie $\star$ algebra on $X$ is a $\mathcal{D}$-module $\mathcal{L}$ on $X$ equipped with a Lie bracket

$$\mathcal{L} \boxtimes \mathcal{L} \to \Delta_! \mathcal{L}.$$ 

**Example:** we have a canonical embedding

$$\mathcal{A}_X \boxtimes \mathcal{A}_X \to j_* j^* (\mathcal{A}_X \boxtimes \mathcal{A}_X).$$

So every chiral algebra $\mathcal{A}_X$ is a Lie $\star$ algebra.
Universal chiral enveloping algebras

The resulting forgetful functor

\[ F : \{ \text{chiral algebras} \} \rightarrow \{ \text{Lie} \star \text{algebras} \} \]

has a left adjoint

\[ U^{\text{ch}} : \{ \text{Lie} \star \text{algebras} \} \rightarrow \{ \text{chiral algebras} \} . \]

\(U^{\text{ch}}(\mathcal{L})\) is the universal chiral envelope of \(\mathcal{L}\).

1. \(U^{\text{ch}}(\mathcal{L})\) has a natural filtration, and there is a version of the PBW theorem.
2. \(U^{\text{ch}}(\mathcal{L})\) has a structure of chiral Hopf algebra.
Commutative chiral algebras

A chiral algebra $A_X$ is **commutative** if the underlying Lie $\star$ bracket is zero.

**Translation into factorisation language:**

\[
\begin{array}{c}
j_*j^* (A_X \boxtimes A_X) \\
\downarrow \sim \\
A_{X^2} \rightarrow j_*j^* (A_{X^2}) \rightarrow \Delta! \Delta^! A_{X^2} \\
\downarrow \sim \\
\Delta! A_X
\end{array}
\]
Commutative chiral algebras

A chiral algebra $\mathcal{A}_X$ is **commutative** if the underlying Lie $\star$ bracket is zero.

Translation into factorisation language:

$$\mathcal{A}_X \otimes \mathcal{A}_X \xrightarrow{j_*j^*} (\mathcal{A}_X \otimes \mathcal{A}_X)$$

$$\mathcal{A}_X^2 \xrightarrow{j_*j^*} (\mathcal{A}_X^2) \xrightarrow{\Delta!\Delta^!} \mathcal{A}_X^2$$

$$\xrightarrow{?} 0 \xrightarrow{?} \Delta!\mathcal{A}_X$$
A chiral algebra $\mathcal{A}_X$ is **commutative** if the underlying Lie $\star$ bracket is zero.

**Translation into factorisation language:**

\[
\begin{array}{c}
\mathcal{A}_X \boxtimes \mathcal{A}_X \longrightarrow j_*j^* (\mathcal{A}_X \boxtimes \mathcal{A}_X) \\
\downarrow \text{?} \\
\mathcal{A}_X^2 \longrightarrow j_*j^* (\mathcal{A}_X^2) \longrightarrow \Delta! \Delta^! \mathcal{A}_X^2 \\
\downarrow \text{?} \\
\Delta! \mathcal{A}_X
\end{array}
\]
Commutative factorisation algebras

A factorisation algebra \( \{ \mathcal{A}_{X^I} \} \) is commutative if every factorisation isomorphism

\[
\left( \alpha \right)^{-1} : j^* \left( \bigotimes_{j \in J} \mathcal{A}_{X^I_j} \right) \xrightarrow{\sim} j^* \mathcal{A}_{X^I}
\]

extends to a map of \( D \)-modules on all of \( X^I \):

\[
\bigotimes_{j \in J} \mathcal{A}_{X^I_j} \to \mathcal{A}_{X^I}.
\]

Proposition (Beilinson–Drinfeld)

We have equivalences of categories

\[
\left\{ \text{commutative factorisation algebras} \right\} \xrightarrow{\sim} \left\{ \text{commutative chiral algebras} \right\} \xrightarrow{\sim} \left\{ \text{commutative } D_{X^I}-\text{algebras} \right\}.
\]
Section 3

Results on $\mathcal{H}_{\text{Ran } X}$
Chiral homology

Let $p_{\text{Ran}X} : \text{Ran} X \to \text{pt}$.

The chiral homology of a factorisation algebra $\mathcal{A}_{\text{Ran}X}$ is defined by

$$\int \mathcal{A}_{\text{Ran}X} := p_{\text{Ran}X}_! \mathcal{A}_{\text{Ran}X}.$$ 

It is a derived formulation of the space of conformal blocks of a vertex algebra $V$:

$$H^0(\int \mathcal{V}_{\text{Ran}X}) = \text{space of conformal blocks of } V.$$
The chiral homology of $\mathcal{H}_{\text{Ran} \ X}$

**Goal:** compute \( \int \mathcal{H}_{\text{Ran} \ X} := p_{\text{Ran} \ X} \cdot ! f ! \omega \mathcal{Hilb}_{\text{Ran} \ X} \).

\[ \Rightarrow \int \mathcal{H}_{\text{Ran} \ X} \simeq p_{\text{Hilb} \ X} \cdot ! \rho ! \omega \mathcal{Hilb}_{\text{Ran} \ X} \simeq p_{\text{Hilb} \ X} \cdot ! \rho ! \rho ! ! \omega \mathcal{Hilb} \ X \cdot. \]
The chiral homology of $\mathcal{H}_{\text{Ran} X}$

**Theorem**

$$\rho ! : \mathcal{D}(\text{Hilb}_X) \to \mathcal{D}(\text{Hilb}_{\text{Ran} X})$$

is fully faithful, and hence $\rho ! \circ \rho ! \to \text{id}_{\mathcal{D}(\text{Hilb}_X)}$ is an equivalence.

**Corollary**

$$\int \mathcal{H}_{\text{Ran} X} \simeq p_{\text{Hilb}_X,!}\omega_{\text{Hilb}_X} := H_{dR}^\bullet(\text{Hilb}_X).$$
The assignment

\[ X \quad \text{dim. } d \quad \mapsto \quad \mathcal{H}_{\text{Ran}} X \]

gives rise to a **universal factorisation algebra** of dimension d.

i.e. it behaves well in families, and is compatible under pullback by \( \acute{\text{e}} \text{tale morphisms} \ Y \rightarrow X \).

This allows us to reduce to the study of \( \mathcal{H}_{\text{Ran}} X \) for \( X = \mathbb{A}^d = \text{Spec } k[x_1, \ldots, x_d] \).
Identifying the factorisation algebra structure on $\mathcal{H}_{\text{Ran} \mathbb{A}^d}$

Conjecture

$\mathcal{H}_{\text{Ran} \mathbb{A}^d}$ is a commutative factorisation algebra.

Remarks on the proof:

1. The case $d = 1$ is clear:
   $\mathcal{Hilb}_{\text{Ran} \mathbb{A}^1}$ is a commutative factorisation space.
2. The case $d = 2$ has been proven by Kotov using deformation theory.
Strategy for general $d$: first step

The choice of a global coordinate system $\{x_1, \ldots, x_d\}$ gives an identification of

$$\text{Hilb}_X, 0 := \{ \xi \in \text{Hilb}_X \mid \text{Supp}(\xi) = \{0\} \}$$

with $\text{Hilb}_{X,p}$ for every $p \in X = \mathbb{A}^d$.

$$\Rightarrow \text{Hilb}_X \simeq X \times \text{Hilb}_{X,0}.$$ 

It follows that

$$\mathcal{H}_X \simeq \omega_X \otimes H_{dR}^\bullet(\text{Hilb}_{X,0}).$$
Strategy for general $d$: second step

Universality of $\mathcal{H}_{\text{Ran}}$ means that, in particular, the fibre of $\mathcal{H}_{\mathbb{A}^d}$ over $0 \in \mathbb{A}^d$, is a representation of the group

$$G = \text{Aut}_k[[t_1, \ldots, t_d]].$$

This fibre is $H^\bullet_{dR}(\text{Hilb}_X, 0)$, and the representation is induced from the action of $G$ on the space $\text{Hilb}_X, 0$. 
Strategy for general $d$: steps 3, 4 . . .

**Claim 1:** The induced action is canonically trivial, except perhaps for an action of $\mathbb{G}_m \subset G$ corresponding to a grading.

**Claim 2:** This forces the chiral bracket

$$j_*j^*(\omega_X \boxtimes \omega_X) \otimes H^\bullet_{dR}(\text{Hilb}_X,0) \otimes H^\bullet_{dR}(\text{Hilb}_X,0) \rightarrow \Delta!(\omega_X) \otimes H^\bullet_{dR}(\text{Hilb}_X,0)$$

to be of the form $\mu_{\omega_X} \otimes m$, where $m$ is a map

$$H^\bullet_{dR}(\text{Hilb}_X,0) \otimes H^\bullet_{dR}(\text{Hilb}_X,0) \rightarrow H^\bullet_{dR}(\text{Hilb}_X,0).$$

**Claim 3:** $m$ induces a commutative $\mathcal{D}_X$-algebra structure on $\mathcal{H}_X = \omega_X \otimes H^\bullet_{dR}(\text{Hilb}_X,0)$.

Claims 1 and 2 seem straightforward to prove in the non-derived setting, but in the derived setting there are subtleties.
Future directions

- Push forward other sheaves to get more interesting factorisation algebras: replace $\omega_{\text{Hilb}_{\chi_1}}$ by sheaves constructed from e.g. tautological bundles, sheaves of vanishing cycles.
- How is this related to the work of Nakajima and Grojnowski?