Robust scale and autocovariance estimation

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School of Mathematics and Statistics
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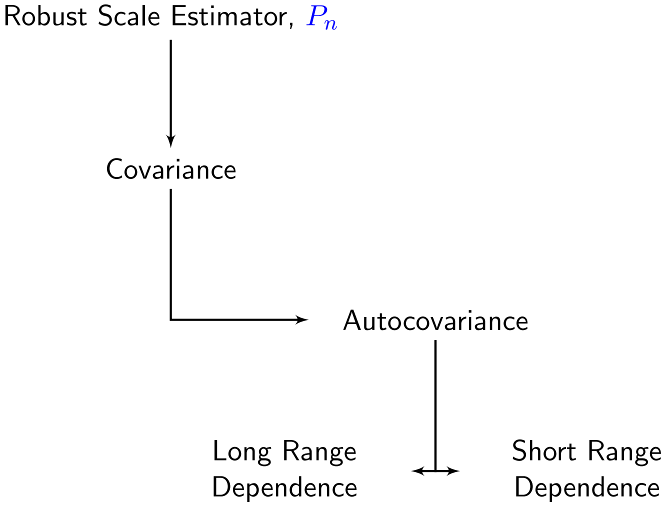
Outline


Robust Scale Estimator, P_n
Covariance
Autocovariance
Long Range Dependence
Short Range Dependence

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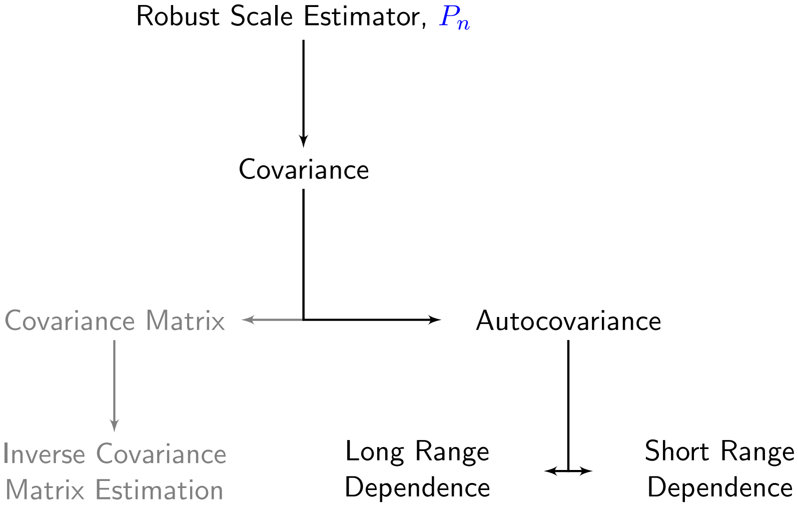
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Covariance Matrix
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Inverse Covariance Matrix Estimation
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The robust scale estimator P_n

Autocovariance estimation using P_n

Conclusion and key references

History of scale efficiencies at the Gaussian ($n = 20$)

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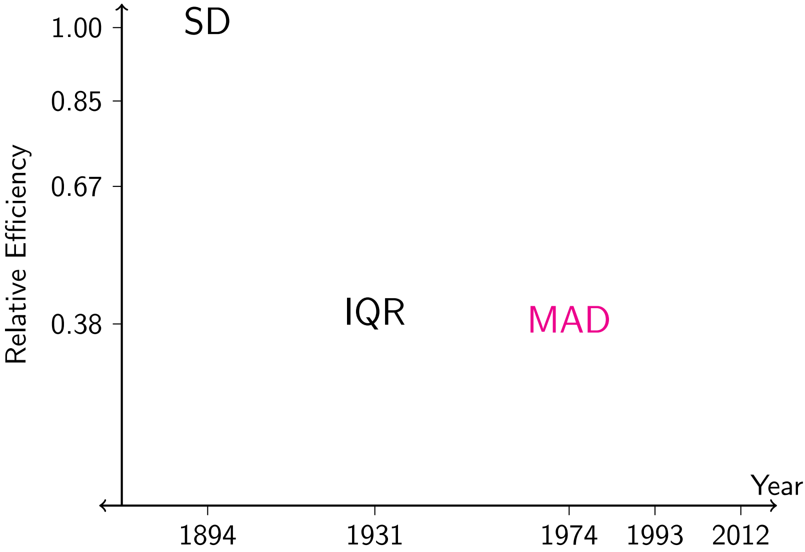

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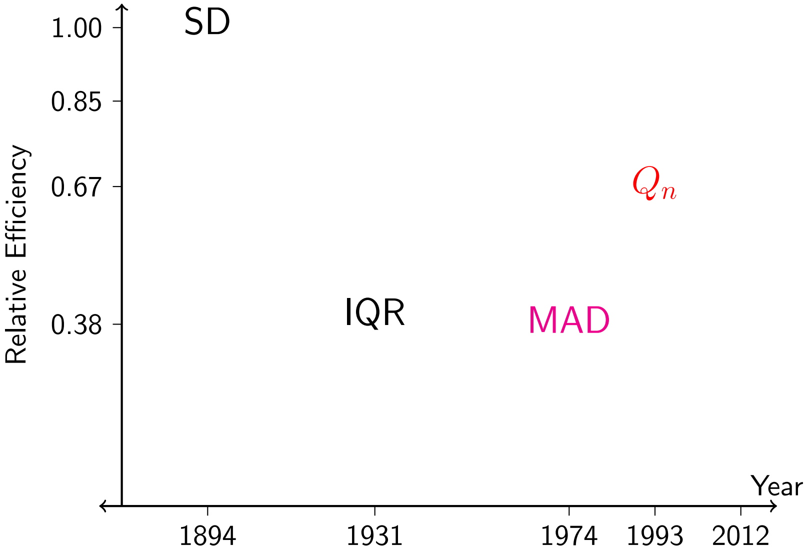

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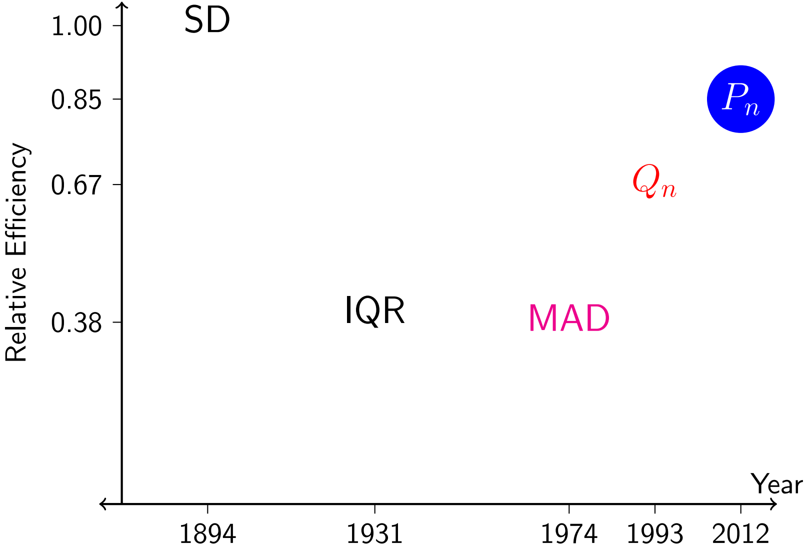

Pairwise mean scale estimator: P_n

- Consider the U-statistic, based on the pairwise mean kernel,

$$U_n(\mathbf{X}) := \binom{n}{2}^{-1} \sum_{i<j} \frac{X_i + X_j}{2}.$$

- Let $H(t) = P((X_i + X_j)/2 \leq t)$ be the cdf of the kernels with corresponding empirical distribution function,

$$H_n(t) := \binom{n}{2}^{-1} \sum_{i<j} \mathbb{I}\left\{\frac{X_i + X_j}{2} \leq t\right\}, \quad \text{for } t \in \mathbb{R}.$$

Definition (Interquartile range of pairwise means)

$$P_n = c\left[H_n^{-1}(0.75) - H_n^{-1}(0.25)\right],$$

where $c \approx 1.048$ is a correction factor to ensure P_n is consistent for the standard deviation when the underlying observations are Gaussian.

Influence curve

- Hampel (1974) defines the influence curve for a functional T at distribution F as
$$\mathrm{IC}(x; T, F) = \lim_{\epsilon \downarrow 0} \frac{T((1-\epsilon)F + \epsilon\delta_x) - T(F)}{\epsilon}$$
where δ_x has all its mass at x.
- Serfling (1984) outlines the IC for GL-statistics.

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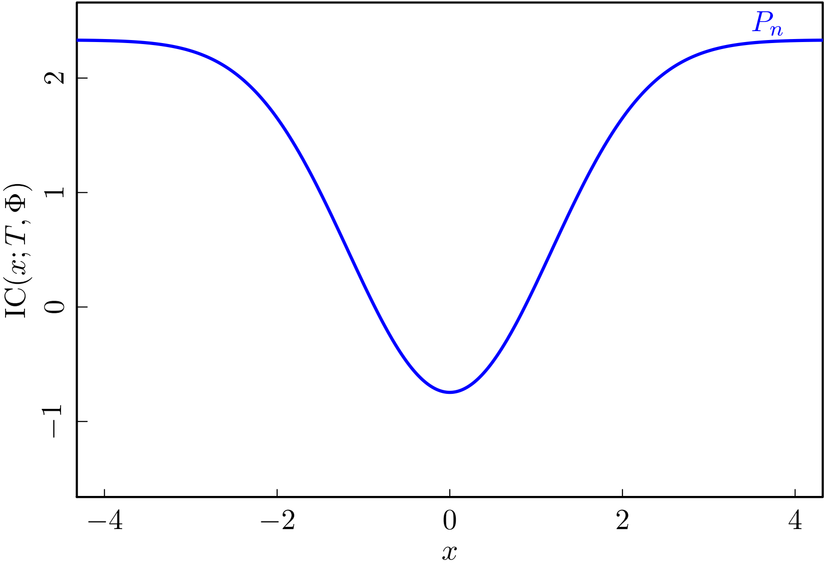
Influence curve for P_n (Tarr, Müller and Weber, 2012)

Assuming that F has derivative $f>0$ on $[F^{-1}(\epsilon),F^{-1}(1-\epsilon)]$ for all $\epsilon>0$,
$$\mathrm{IC}(x;P,F)=c\left[\frac{0.75-F(2H_F^{-1}(0.75)-x)}{\int f(2H_F^{-1}(0.75)-x)f(x)\,\mathrm{d}\,x}-\frac{0.25-F(2H_F^{-1}(0.25)-x)}{\int f(2H_F^{-1}(0.25)-x)f(x)\,\mathrm{d}\,x}\right].$$

Influence curves when $F = \Phi$



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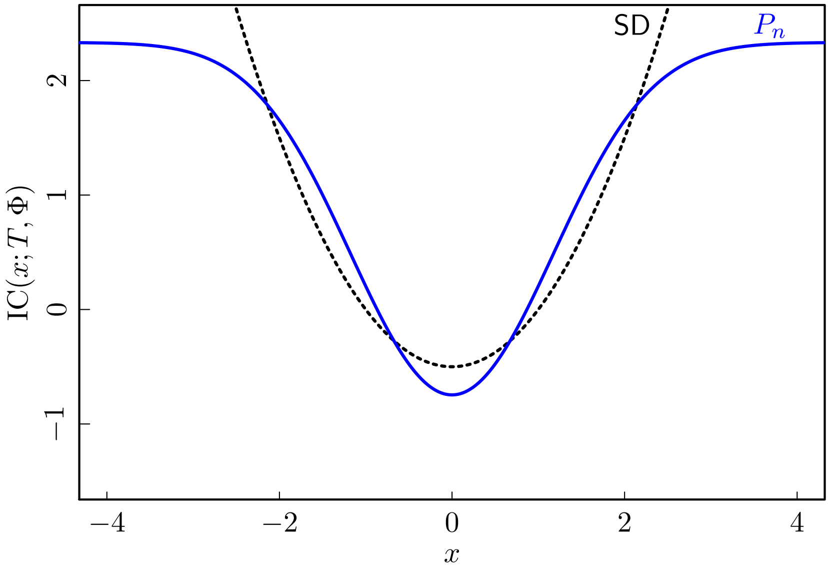

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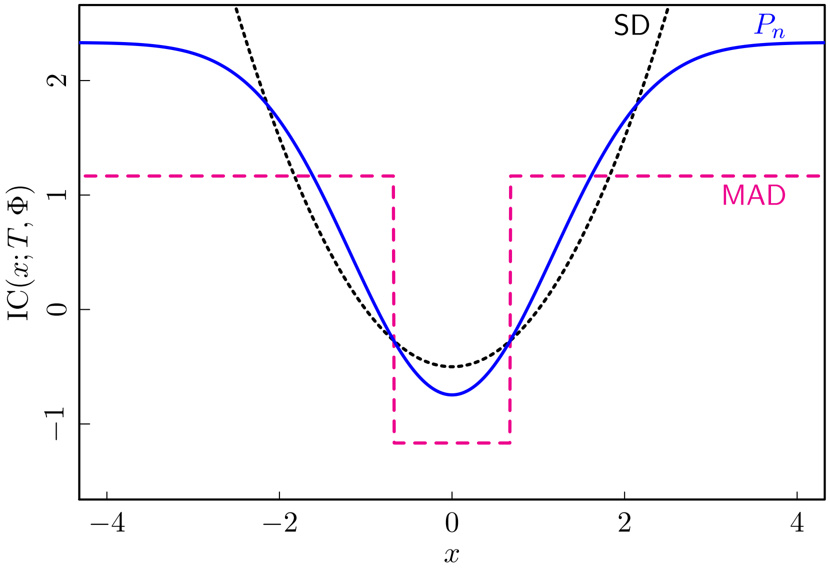

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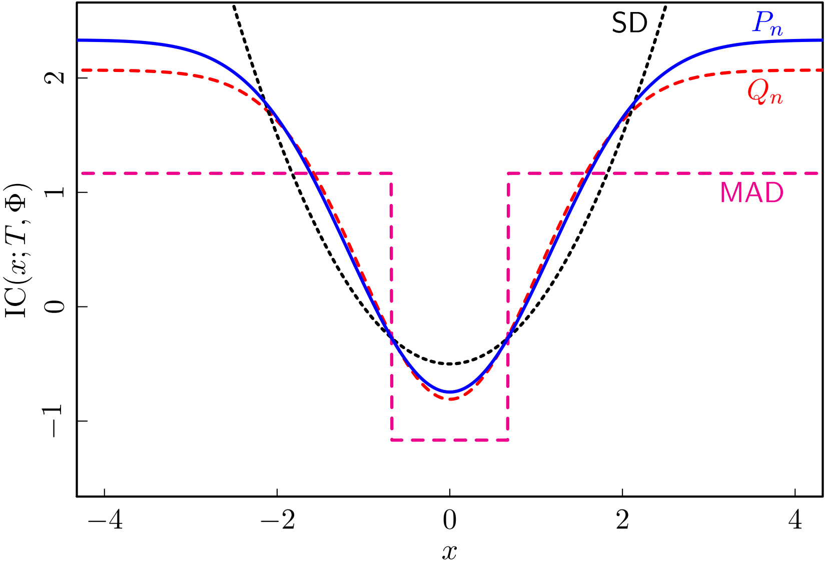

Asymptotic variance and relative efficiency

- Tarr, Müller and Weber (2012) show that when the underlying observations are independent, P_n is asymptotically normal with variance, V, given by the expected square of the influence function.
- When the underlying data are independent Gaussian,

$$V(P_n, \Phi) = \int \mathrm{IC}(x; P, \Phi)^2 \, \mathrm{d}\, \Phi(x) = 0.579.$$

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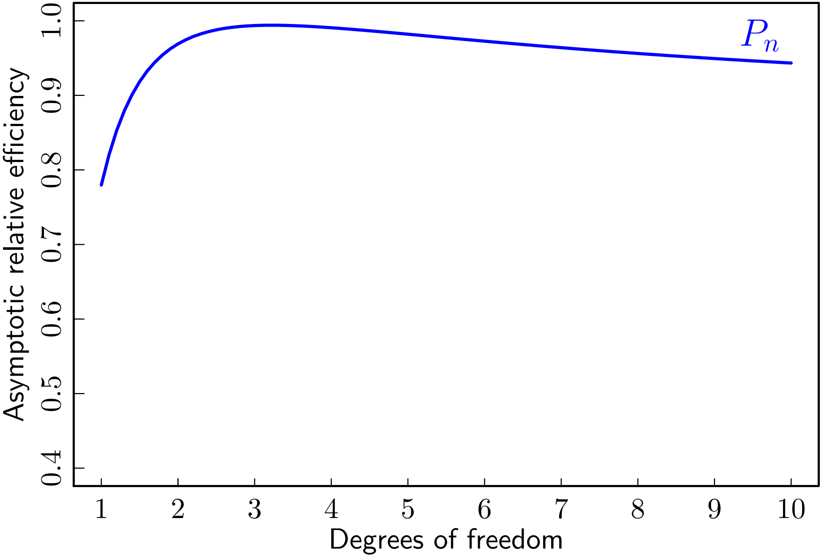
- This equates to an asymptotic efficiency of **0.86** as compared with **0.82** for Q_n and **0.37** for the MAD.

! But how does it compare at heavier tailed distributions?

Asymptotic relative efficiency when $f = t_\nu$ for $\nu \in [1, 10]$



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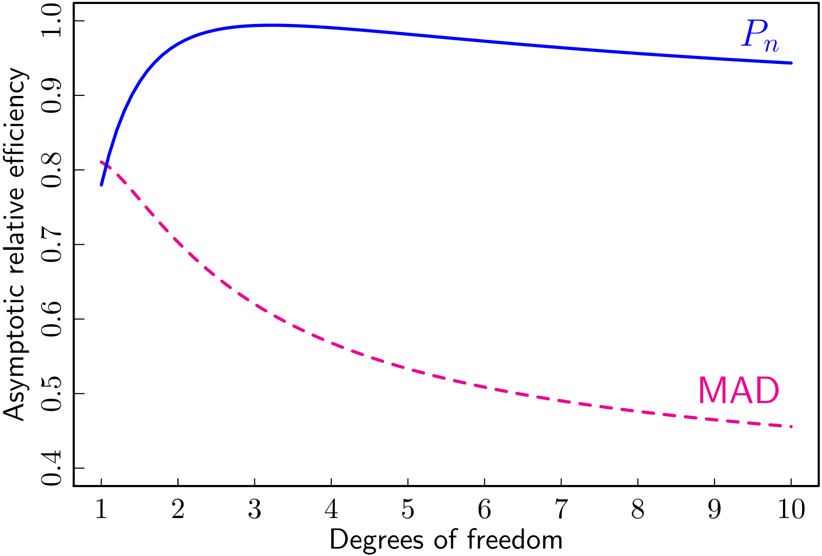

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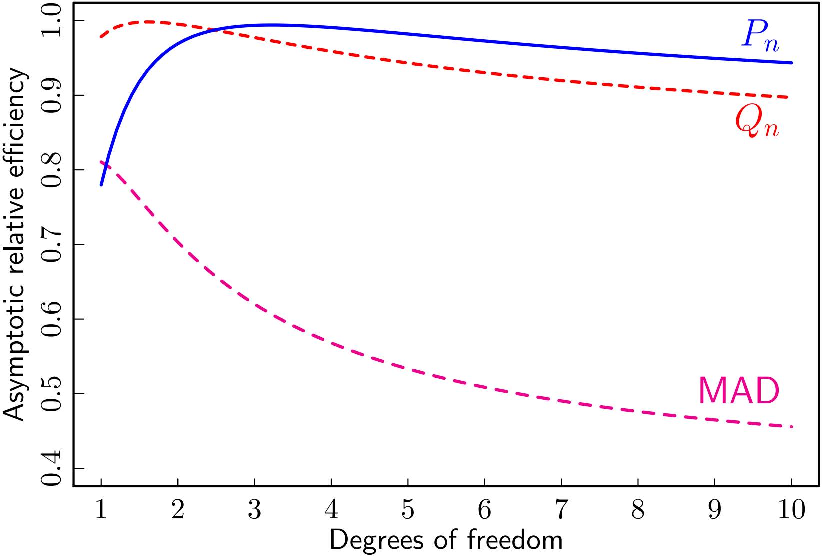

Asymptotic distribution under short range dependence

Assumption 1: Gaussian SRD

Let $\{X_i\}_{i\geq 1}$ be a stationary mean-zero Gaussian process with autocovariances $\gamma(h) = \mathbb{E}(X_1 X_{h+1})$ satisfying $\sum_{h\geq 1} |\gamma(h)| < \infty$.

Result

Under Assumption 1, it can be shown that,

$$\sqrt{n}(P_n - \sigma) = \frac{c_\Phi}{\sqrt{n}} \sum_{i=1}^{n} \mathrm{IC}(X_i, P, \Phi) + o_p(1),$$

hence applying a limit theorem from Arcones (1994) we have

$$\sqrt{n}(P_n - \sigma) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tilde{\sigma}^2),$$

where $\sigma = \sqrt{\gamma(0)}$ and

$$\tilde{\sigma}^2 = \mathbb{E}\left(\mathrm{IC}^2(X_1, P, \Phi)\right) + 2\sum_{k\geq 1} \mathbb{E}\left(\mathrm{IC}(X_1, P, \Phi)\mathrm{IC}(X_{k+1}, P, \Phi)\right).$$

Asymptotic distribution under long range dependence

Assumption 2: Gaussian LRD

Let $\{X_i\}_{i\geq 1}$ be a stationary mean-zero Gaussian process with autocovariance sequence $\gamma(h) = \mathbb{E}(X_1 X_{h+1})$ satisfying $\gamma(h) = h^{-D}L(h)$, $0 < D < 1$, where L is slowly varying at infinity.

Result

Under Assumption 2,

1. If $D > 1/2$,

$$\sqrt{n}(P_n - \sigma) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tilde{\sigma}^2),$$

$$\tilde{\sigma}^2 = \mathbb{E}\,\mathrm{IC}^2(X_1, P, \Phi) + 2\sum_{k\geq 1} \mathbb{E}\,\mathrm{IC}(X_1, P, \Phi)\mathrm{IC}(X_{k+1}, P, \Phi).$$

! Note that this is the same as the SRD result, however the proof is somewhat more involved.

Asymptotic distribution under long range dependence

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Conjecture

Under Assumption 2,

2. If $D < 1/2$,

$$k(D)n^D L(n)^{-1}(P_n - \sigma) \xrightarrow{\mathcal{D}} \frac{\sigma}{2}\left(Z_{2,D}(1) - Z_{1,D}^2(1)\right),$$

where $k(D) = B((1-D)/2, D)$ and B denotes the beta function $Z_{1,D}$ is the standard fBm process and $Z_{2,D}$ is the Rosenblatt process.

! This is consistent with results for other scale estimators, e.g. SD and Q_n (Lévy-Leduc et. al., 2011) and the IQR...

Interquartile range result

Consider equivalent result for the interquartile range,

$$T_n(\mathbf{x}) = c_\Phi \left(F_n^{-1}(3/4) - F_n^{-1}(1/4) \right).$$

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Result

Under Assumption 2 with $D < 1/2$, T_n satisfies the following limit theorem as $n \to \infty$:

$$k(D)n^D L(n)^{-1}(T_n - \sigma) \xrightarrow{\mathcal{D}} \frac{\sigma}{2}\left(Z_{2,D}(1) - Z_{1,D}^2(1)\right).$$

- In establishing this result we used Wu's (2005) Bahadur representation for sample quantiles under LRD:

$$F_n^{-1}(p) - r = \frac{p - F_n(r)}{\phi(r)} + \frac{\bar{X}_n^2}{2}\frac{\phi'(r)}{\phi(r)} + O(n^{h(D)}L_1(n)).$$

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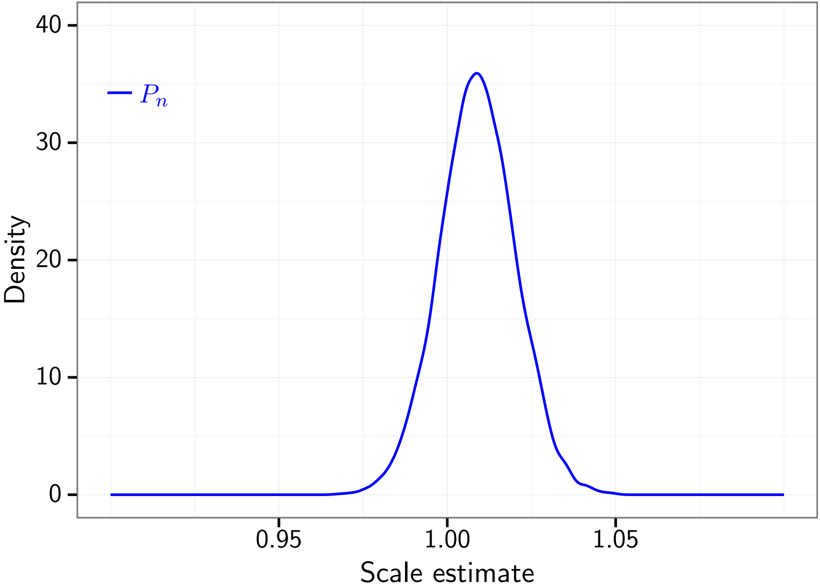
! It appears to be non-trivial to extend this result to the pairwise mean empirical distribution function.

Empirical densities

ARFIMA(0, 0.1, 0); $D = 0.8$; $n = 5000$



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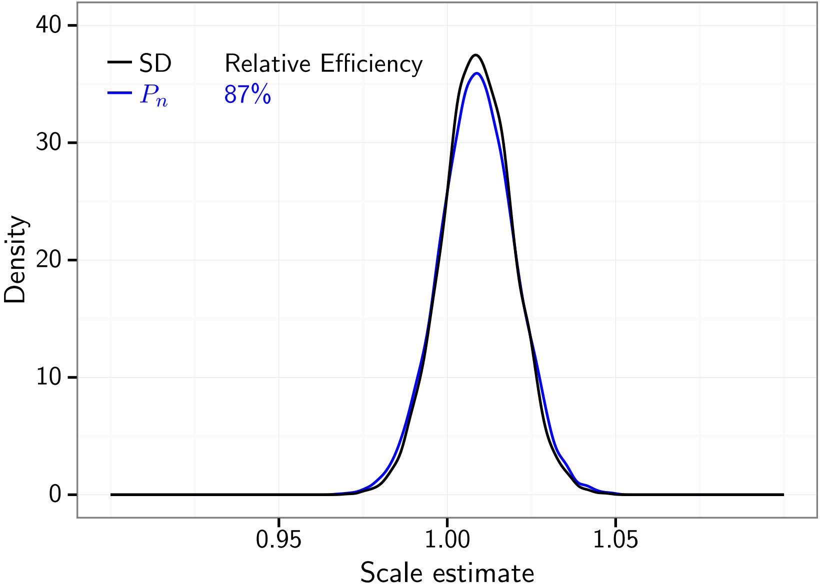

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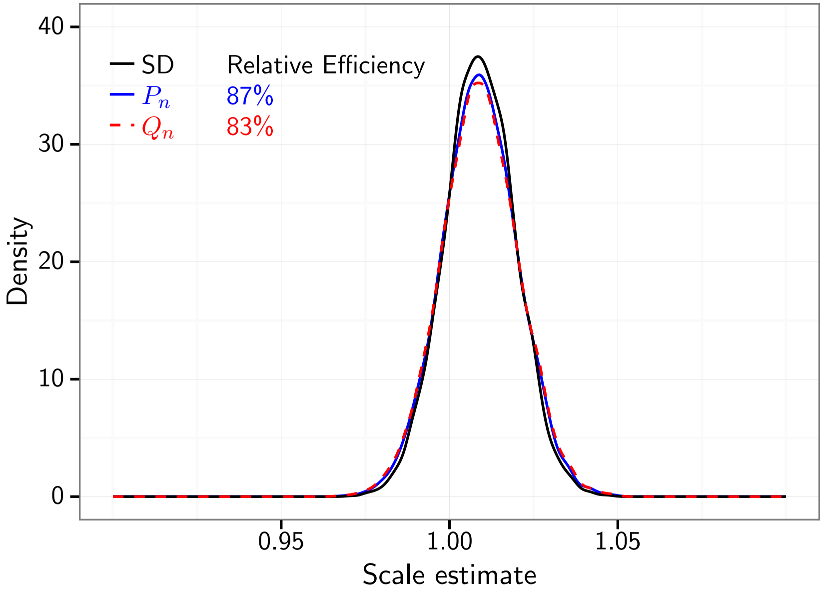

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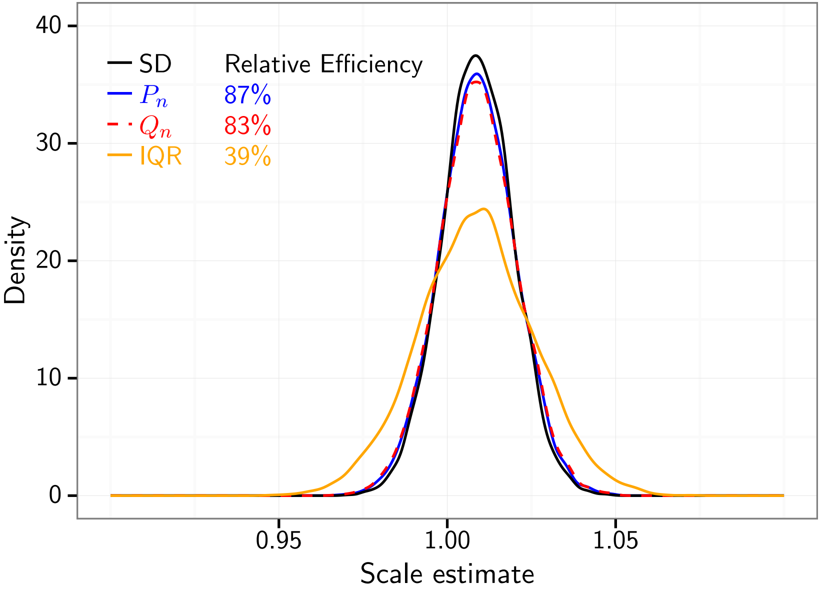

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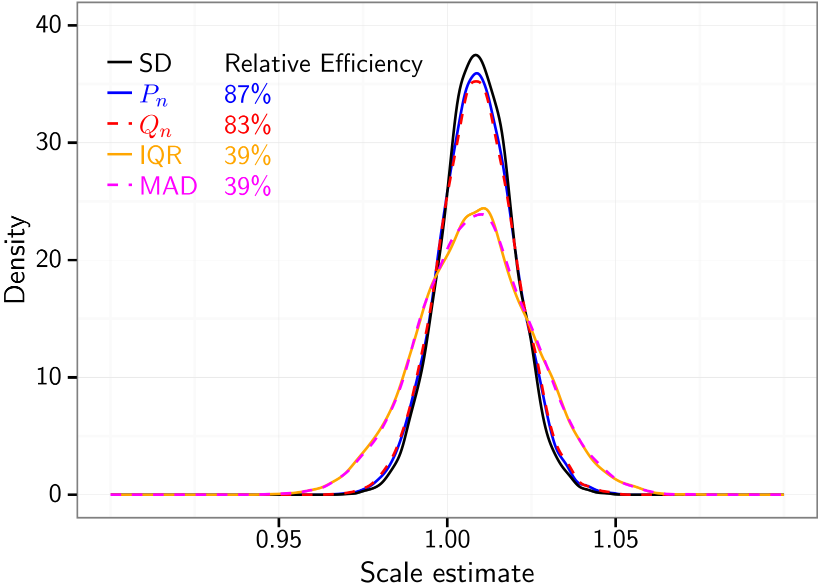

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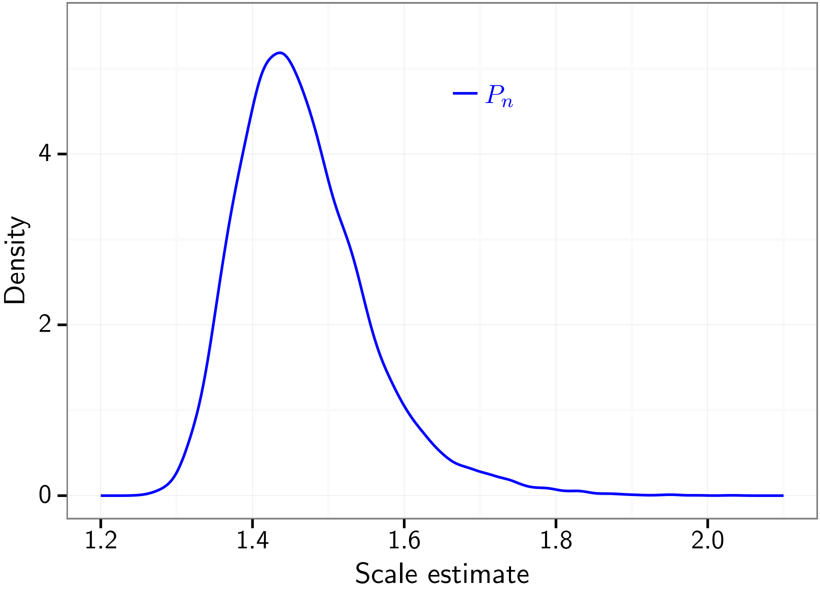

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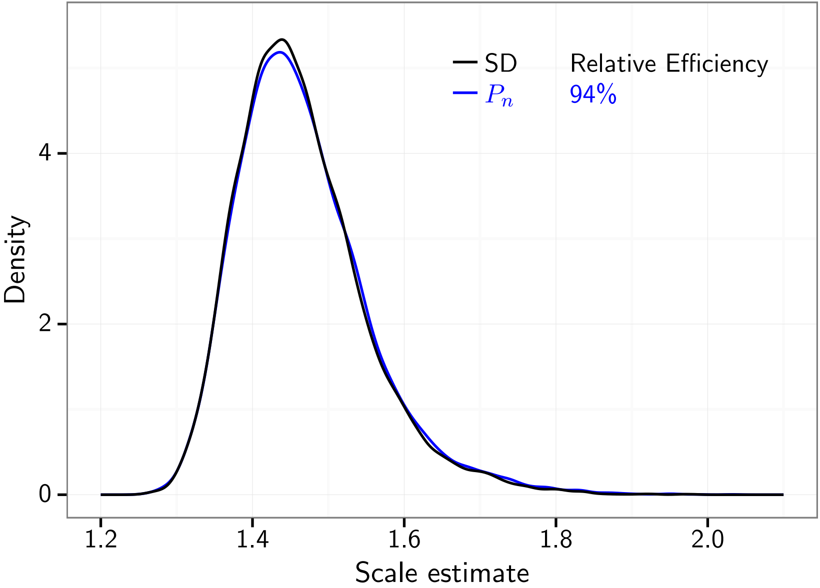

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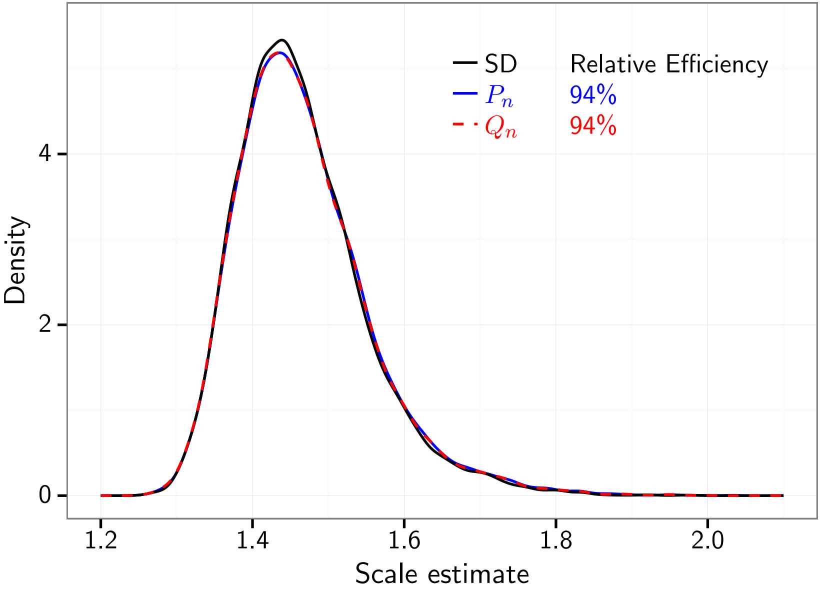

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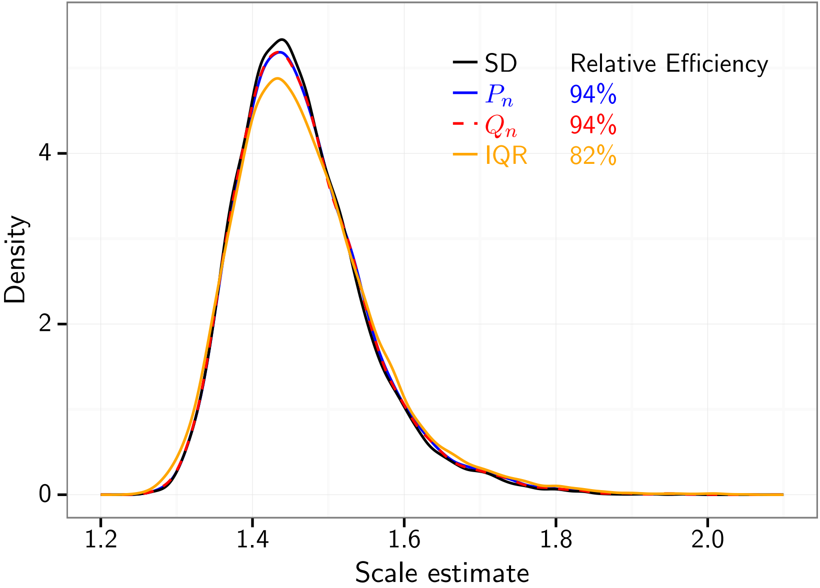

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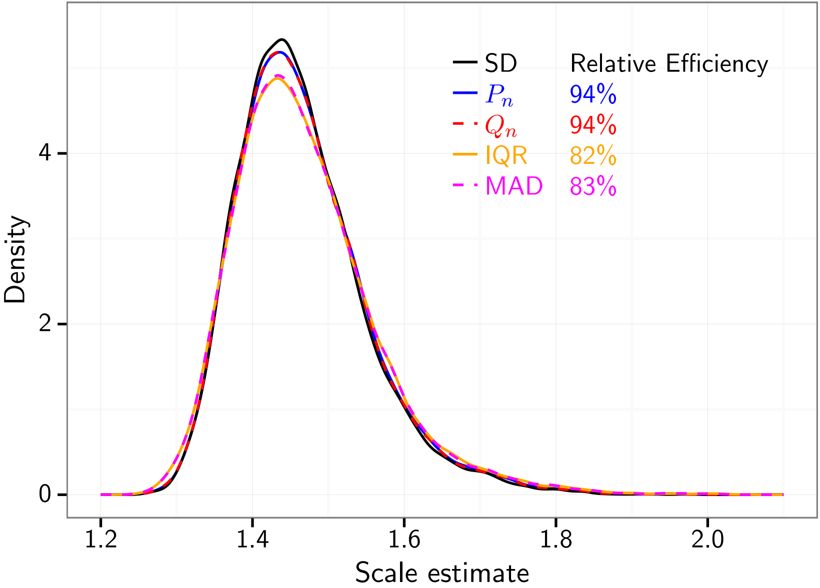
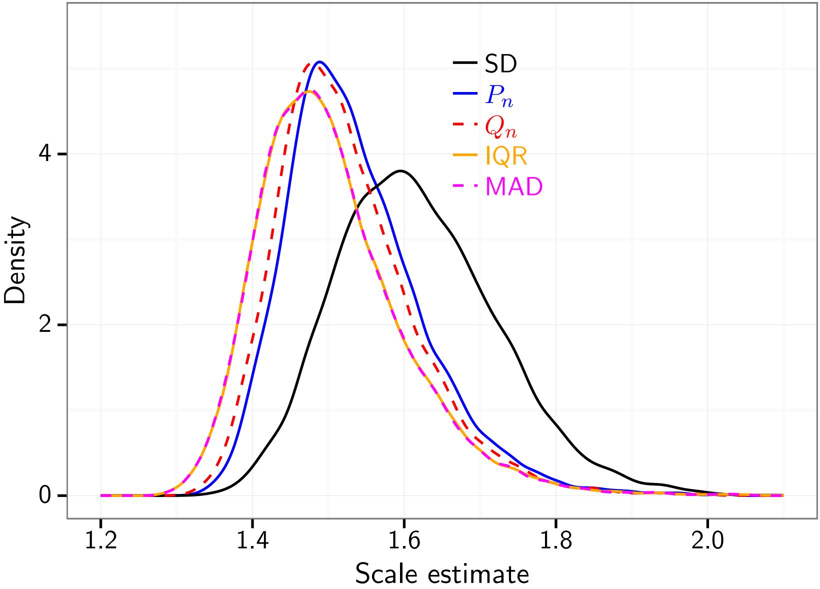
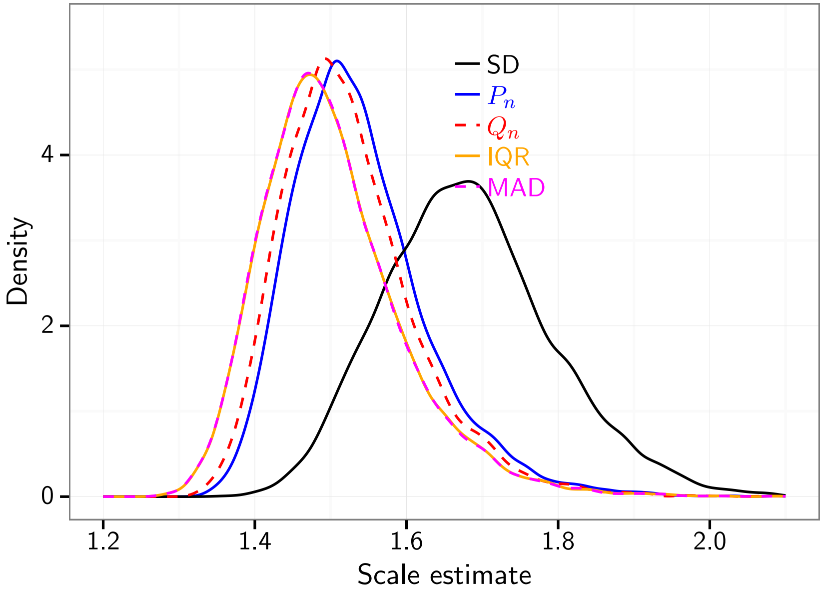


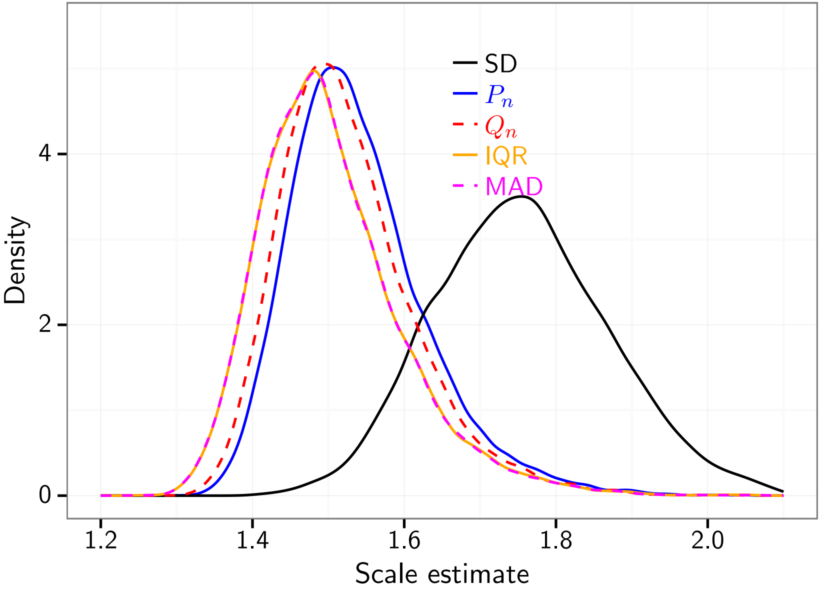
Empirical densities with 2% contamination at 5



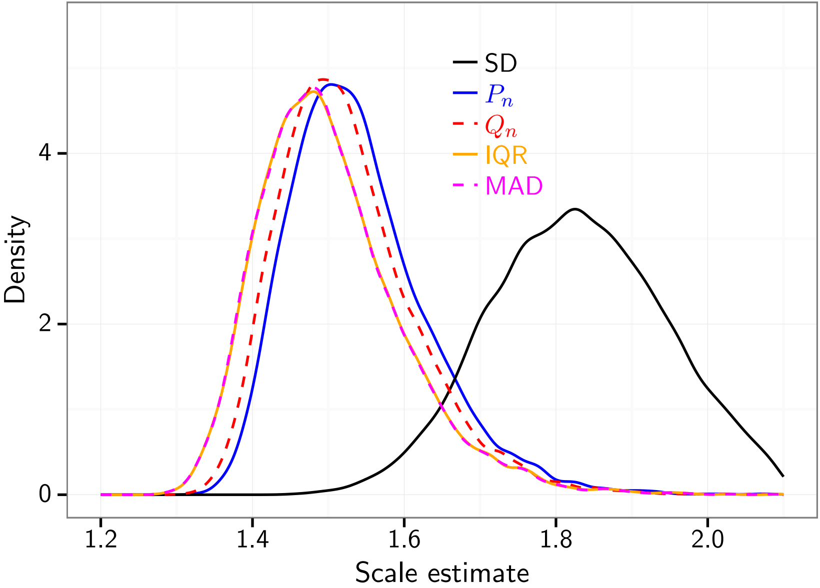
Empirical densities with 2% contamination at 6



Empirical densities with 2% contamination at 7



Empirical densities with 2% contamination at 8



Outline

The robust scale estimator P_n

Autocovariance estimation using P_n

Conclusion and key references

From scale to autocovariance

- Gnanadesikan and Kettenring (1972) highlighted an identity that relates scale and covariance.
- In the autocovariance setting,

$$\gamma(h) = \operatorname{cov}(X_1, X_{h+1}) = \frac{1}{4}\left[\operatorname{var}(X_1 + X_{h+1}) - \operatorname{var}(X_1 - X_{h+1})\right].$$

- For a series of n observations, $\mathbf{X}_n = \{X_t\}_{1 \leq t \leq n}$,

$$\hat{\gamma}_P(h) = \frac{1}{4}\left[P^2_{n-h}(\mathbf{X}_{1:n-h} + \mathbf{X}_{h+1:n}) - P^2_{n-h}(\mathbf{X}_{1:n-h} - \mathbf{X}_{h+1:n})\right].$$

  where $\mathbf{X}_{1:n-h}$ are the first $n-h$ observations in $\mathbf{X}_n$ and $\mathbf{X}_{h+1:n}$ are the last $n-h$ observations.
- The same technique can be used to turn other scale estimators into covariance and correlation estimators, e.g. Ma and Genton (2000) study γ_Q based on Q_n.

Robustness properties

- Following Genton and Ma (1999), we have shown that **influence curve**, $\mathrm{IC}((x,y),\gamma_P,\Phi)$, and therefore the **gross error sensitivity** of γ_P can be derived from the IC of P_n.

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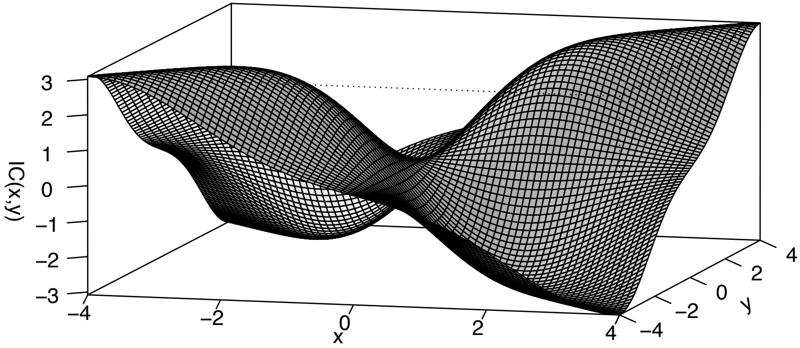

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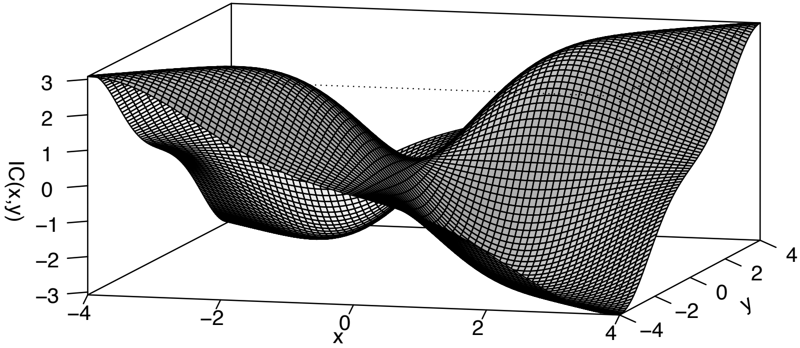

! IC is **bounded** and hence the gross error sensitivity is **finite**.

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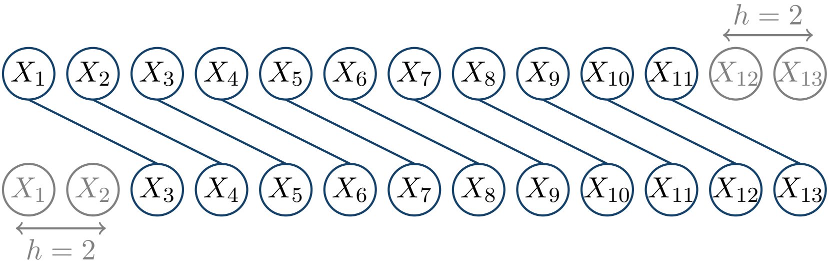

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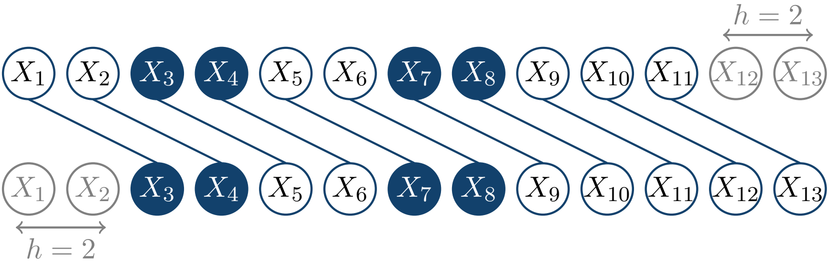
- 4 contaminated observations, denoted by ●



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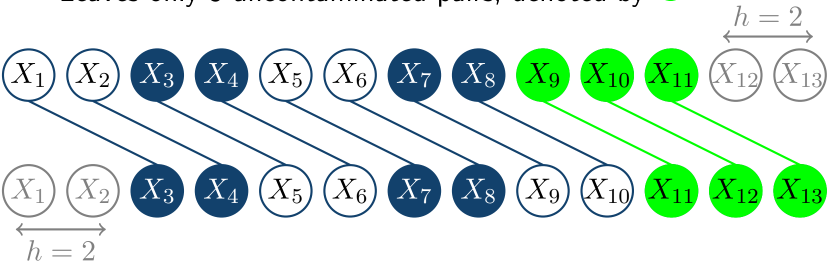
- Leaves only 3 uncontaminated pairs, denoted by ●



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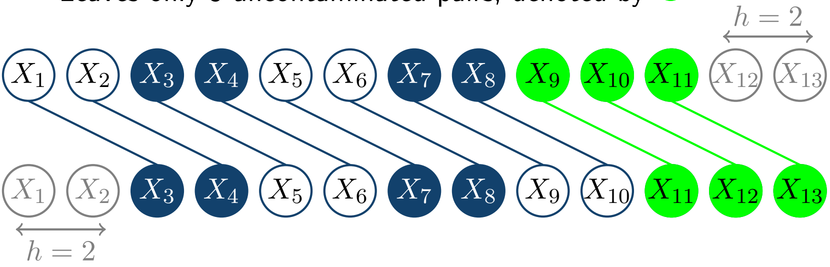


! $\hat{\gamma}_P(h)$ has roughly half the **13.4%** breakdown value of P_n.

Asymptotic distribution under short range dependence

Result

If $\{X_i\}_{i\geq 1}$ satisfy Assumption 1 (SRD) then the sequences $\{X_i + X_{i+h}\}_{i\geq 1}$ and $\{X_i - X_{i+h}\}_{i\geq 1}$ also satisfy Assumption 1, hence we can use the previous results and apply the functional delta method to show,

$$\sqrt{n}(\hat{\gamma}_P(h) - \gamma(h)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \breve{\sigma}^2(h)),$$

where

$$\begin{aligned}\breve{\sigma}^2(h) &= \mathbb{E}\left[\mathrm{IC}^2((X_1, X_{1+h}), \gamma_P, \Phi)\right] \\ &\quad + 2\sum_{k\geq 1} \mathbb{E}\left[\mathrm{IC}((X_1, X_{1+h}), \gamma_P, \Phi)\,\mathrm{IC}((X_{k+1}, X_{k+1+h}), \gamma_P, \Phi)\right].\end{aligned}$$

Asymptotic distribution under long range dependence

Result

Under Assumption 2 (LRD),

1. $D > 1/2$ (same limiting distribution as SRD):

$$\sqrt{n}(\hat{\gamma}_P(h) - \gamma(h)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \breve{\sigma}^2(h)).$$

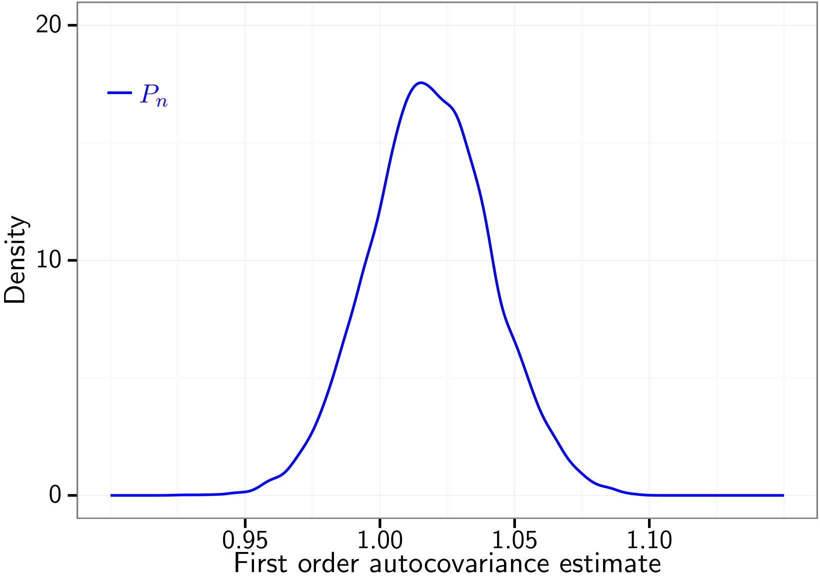
2. $D < 1/2$ (based on conjectured result):

$$\frac{k(D)n^D}{\tilde{L}(n)}(\hat{\gamma}_P(h) - \gamma(h)) \xrightarrow{\mathcal{D}} \frac{\gamma(0) + \gamma(h)}{2}(Z_{2,D}(1) - Z_{1,D}^2(1)),$$

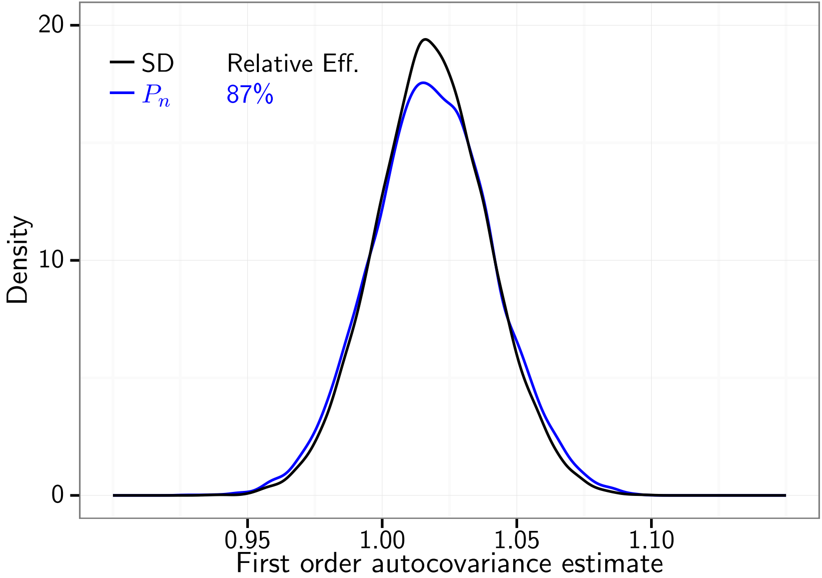
where $k(D) = \mathrm{Beta}((1-D)/2, D)$, $Z_{1,D}$ is the standard fBm process, $Z_{2,D}$ is the Rosenblatt process and

$$\tilde{L}(n) = 2L(n) + L(n+h)(1+h/n)^{-D} + L(n-h)(1-h/n)^{-D}.$$

Empirical densities $\hat{\gamma}_\bullet(1)$ ARFIMA(0, 0.1, 0); $D = 0.8$; $n = 5000$



Empirical densities $\hat{\gamma}_{\bullet}(1)$ ARFIMA(0, 0.1, 0); $D = 0.8$; $n = 5000$



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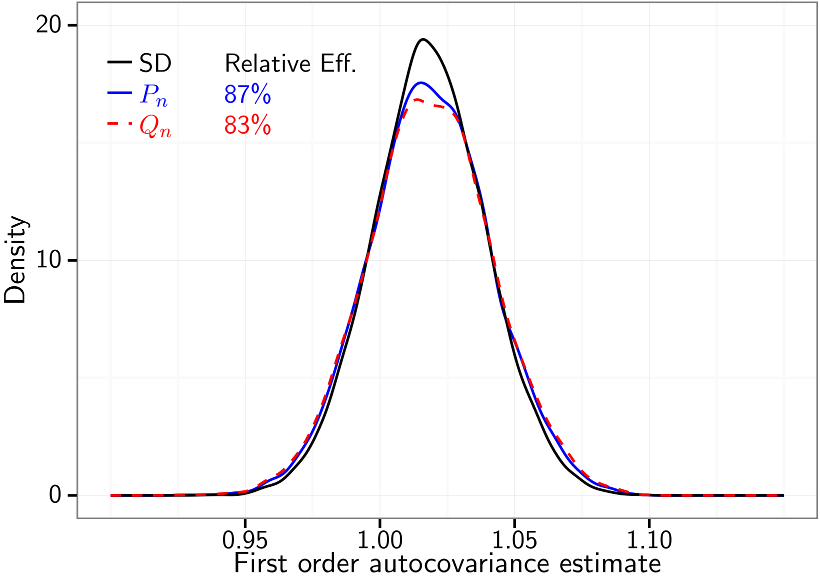
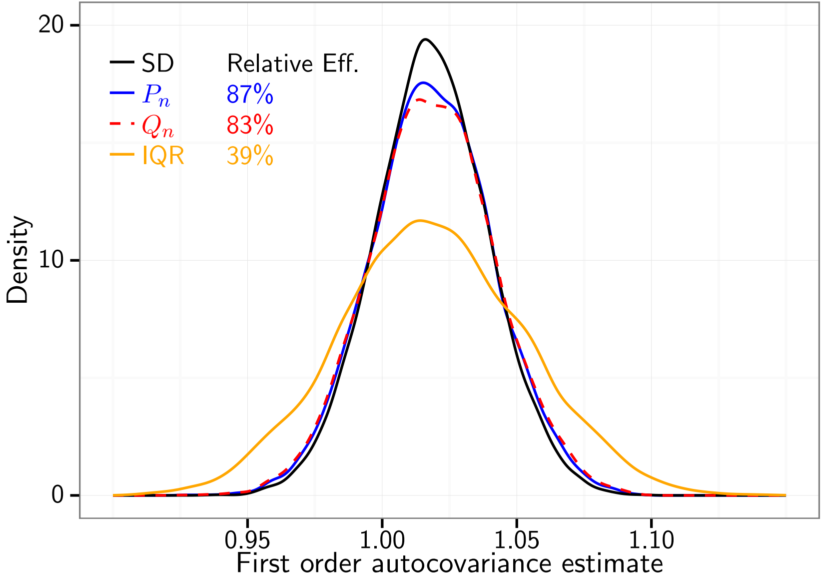
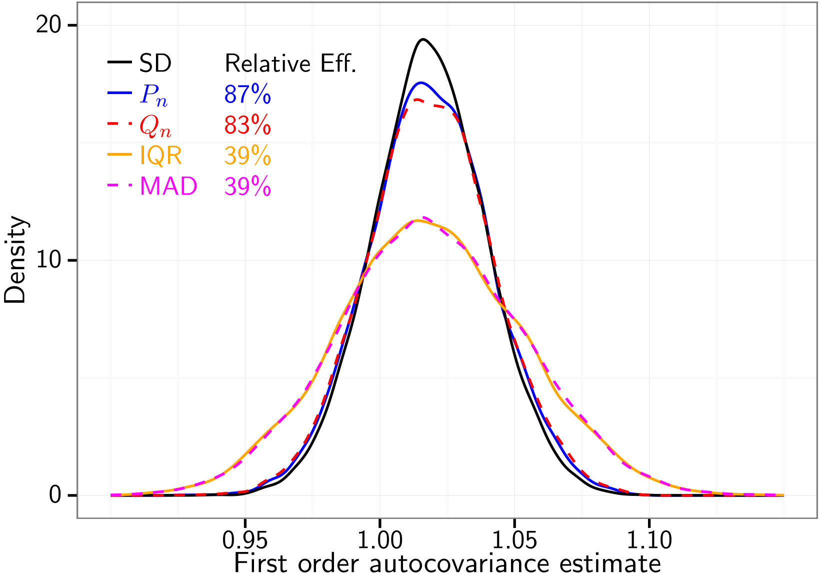


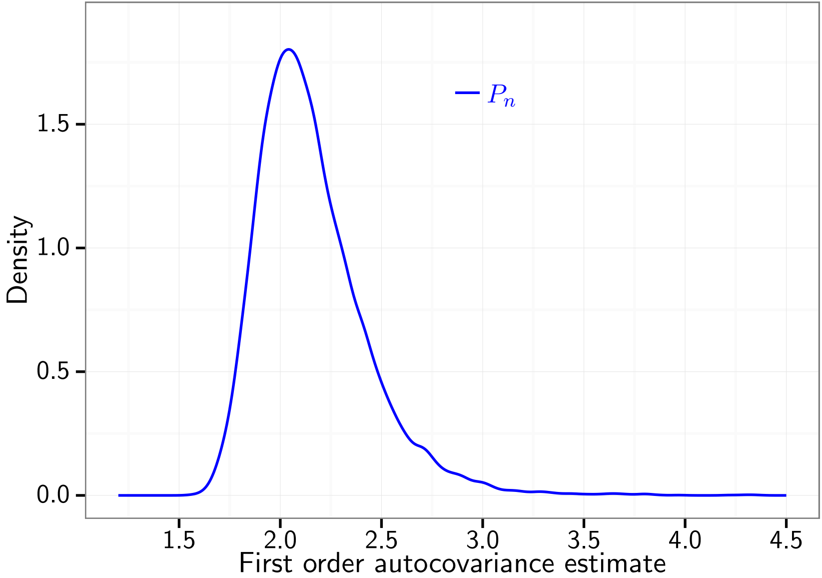
Empirical densities $\hat{\gamma}_{\bullet}(1)$ ARFIMA(0, 0.1, 0); $D = 0.8$; $n = 5000$



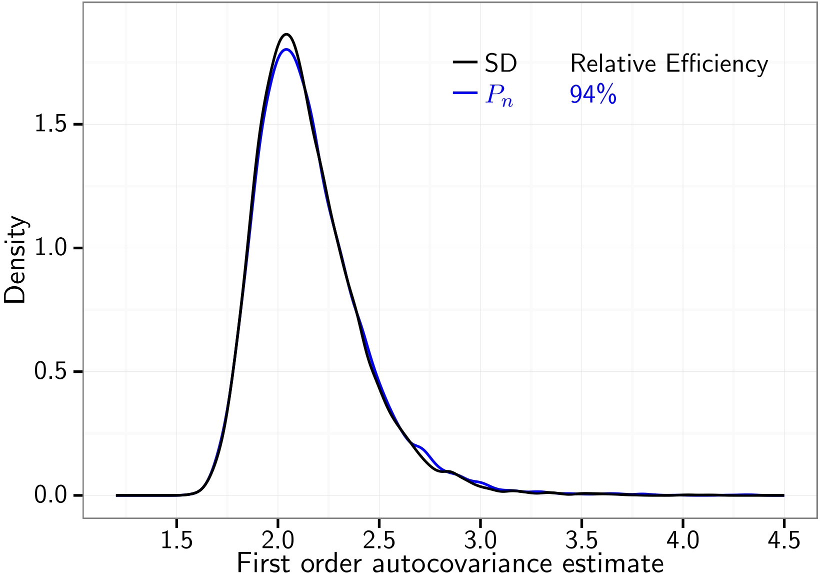
Empirical densities $\hat{\gamma}_{\bullet}(1)$ ARFIMA(0, 0.1, 0); $D = 0.8$; $n = 5000$



Empirical densities $\hat{\gamma}_\bullet(1)$ ARFIMA(0, 0.45, 0); $D = 0.1$; $n = 5000$



Empirical densities $\hat{\gamma}_{\bullet}(1)$ ARFIMA(0, 0.45, 0); $D = 0.1$; $n = 5000$



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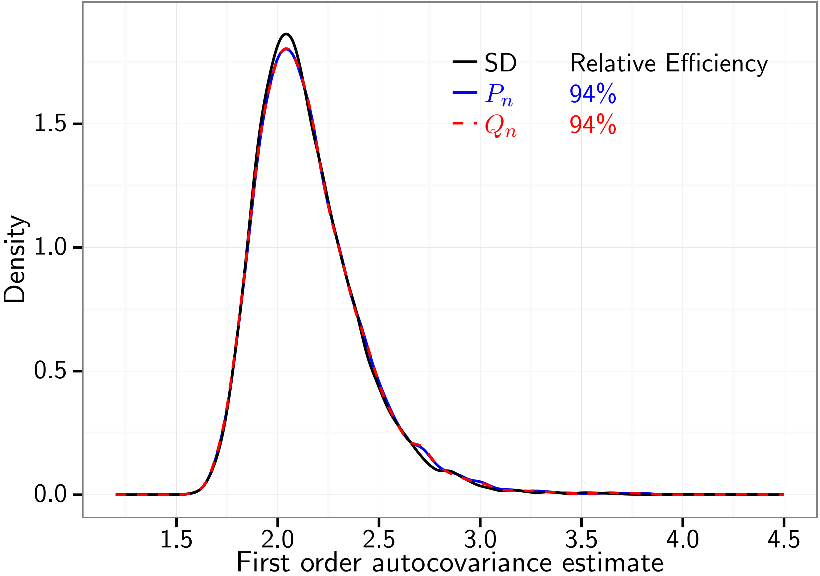
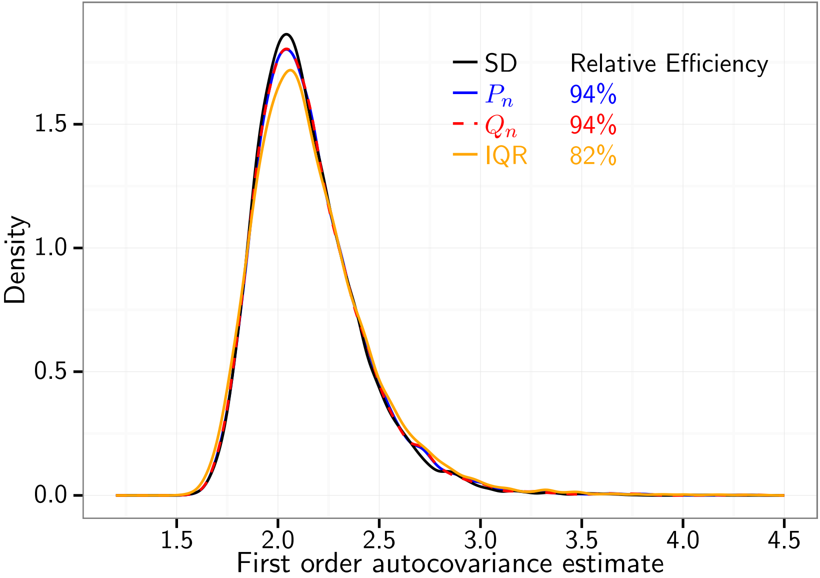
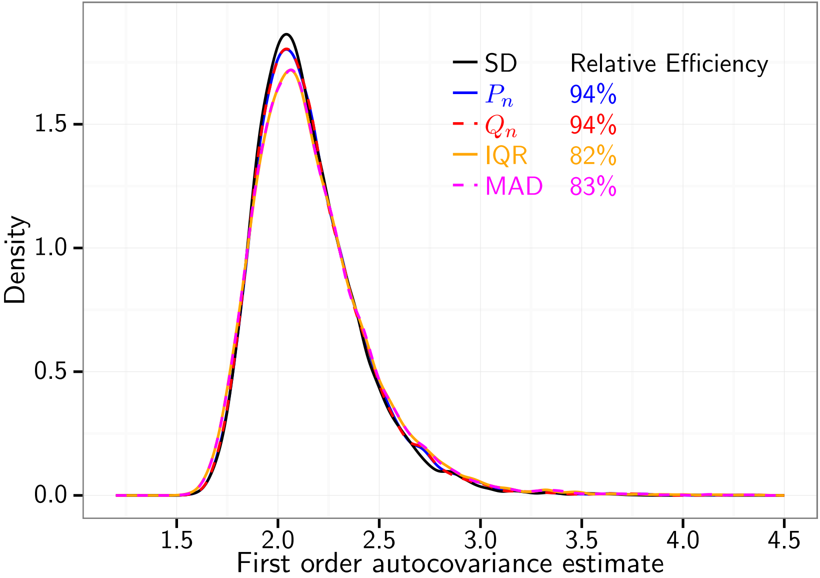


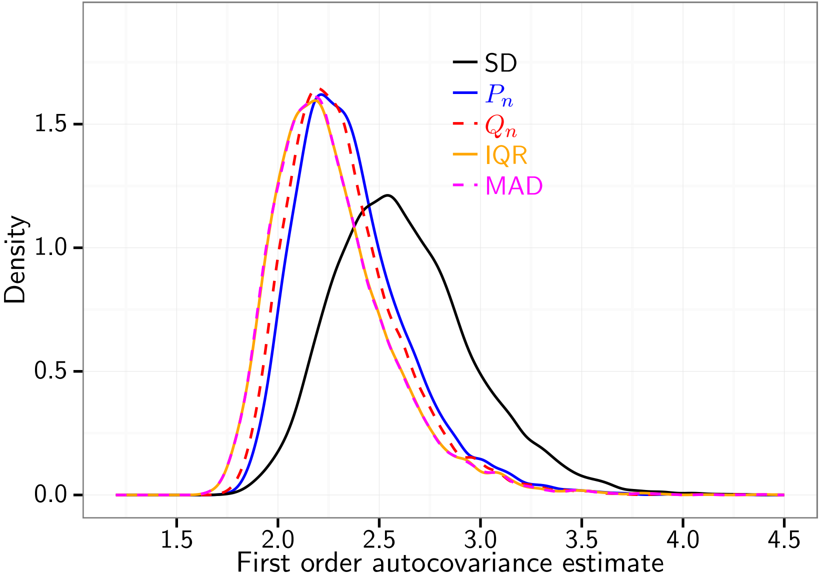
Empirical densities $\hat{\gamma}_{\bullet}(1)$ ARFIMA(0, 0.45, 0); $D = 0.1$; $n = 5000$



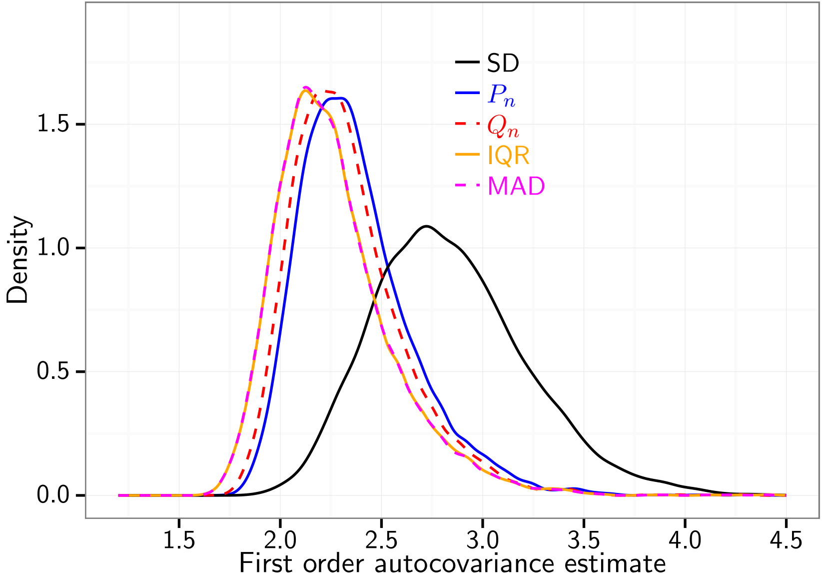
Empirical densities $\hat{\gamma}_\bullet(1)$ ARFIMA(0, 0.45, 0); $D = 0.1$; $n = 5000$



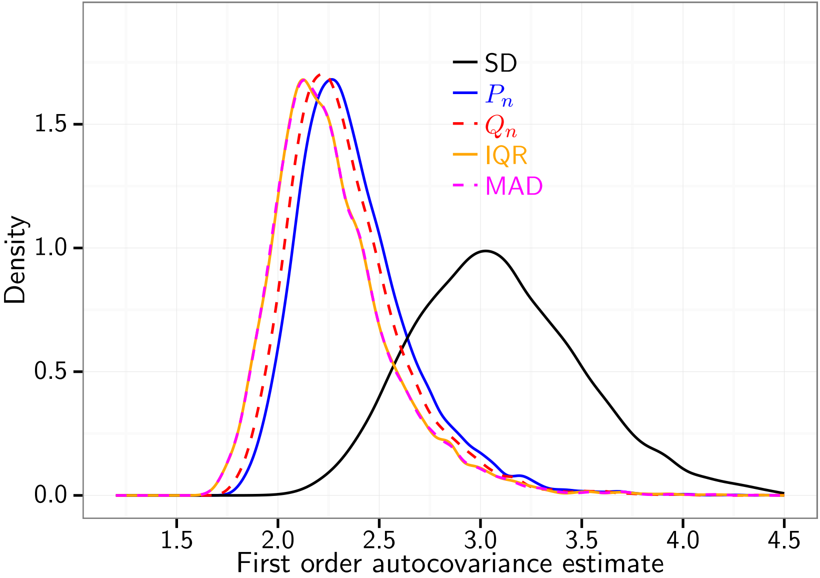
Empirical densities $\hat{\gamma}_{\bullet}(1)$ with 2% contamination at 5



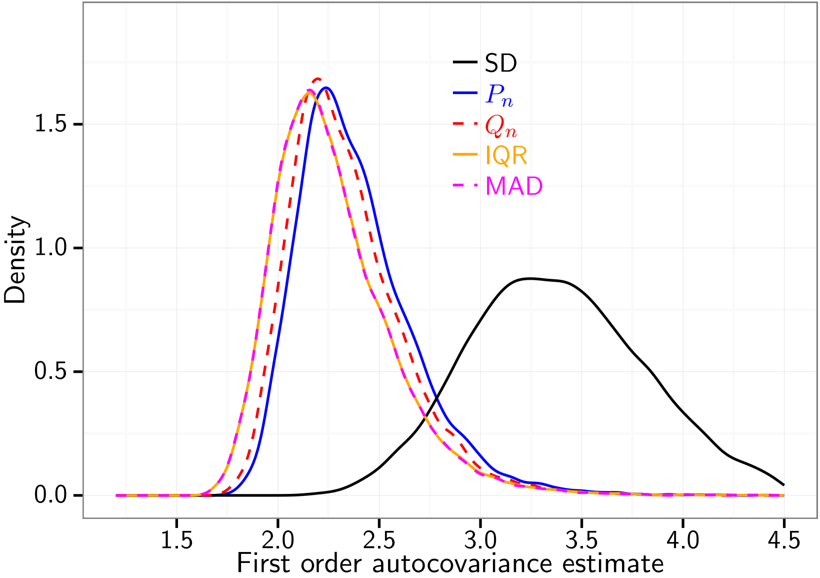
Empirical densities $\hat{\gamma}_\bullet(1)$ with 2% contamination at 6



Empirical densities $\hat{\gamma}_{\bullet}(1)$ with 2% contamination at 7



Empirical densities $\hat{\gamma}_\bullet(1)$ with 2% contamination at 8



Outline

The robust scale estimator P_n

Autocovariance estimation using P_n

Conclusion and key references

Summary

1. Aim

- Efficient and robust scale and autocovariance estimation.

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2. Method

- P_n scale estimator transformed using the GK identity.

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- P_n scale estimator transformed using the GK identity.

3. Results

- 86% asymptotic efficiency at the Gaussian and high asymptotic efficiency at heavier tailed distributions.
- Robustness properties transfer from P_n to γ_P.
- In certain LRD settings robust estimators have very high efficiencies relative to the standard deviation.

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