Hard Lefschetz and the shape of Bruhat intervals

§0. References.

1. Björner & Ekedahl "On the shape of Bruhat intervals" (2009)
2. Achar "Perverse sheaves and applications to RT" (2021 draft)
3. The Stacks project.

§1. The étale setting

Let X variety over $\mathbb{F} = \mathbb{F}_q$ or $\mathbb{k} = \overline{\mathbb{k}}$

$$a_X : X \longrightarrow \operatorname{Spec} \mathbb{F}.$$

$\mathbb{F}'/\mathbb{F}$ a $\mathbb{F}'$-point is a map

$$x : \operatorname{Spec} \mathbb{F}' \longrightarrow X.$$

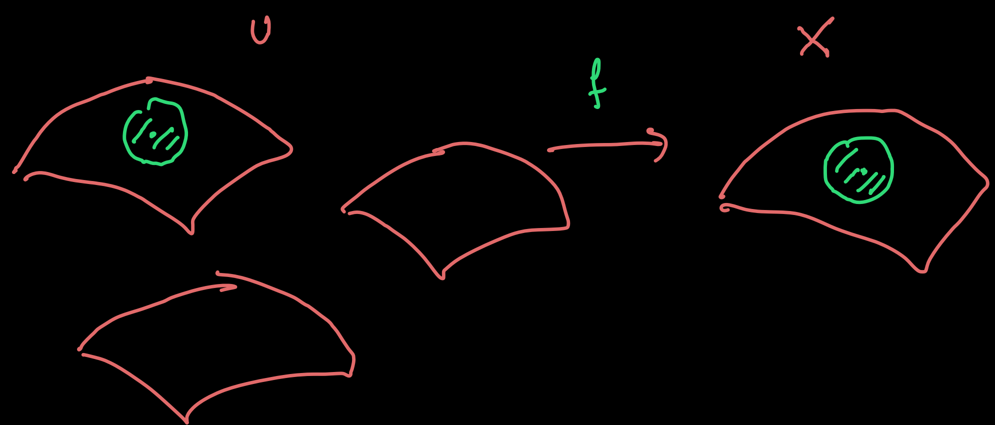
We consider étale maps

$$U \xrightarrow{f} X$$

unramified & flat. They form the the étale topology of X

For X smooth proj var $/\mathbb{C}$ étale $U \to X$ means

- $U = \coprod X_\alpha$ $\quad X_\alpha$ smooth varieties
- $U \to X$ is locally an analytic isomorphism



e.g.

$$\mathrm{Spec}(\mathbb{C}[t]) \longrightarrow \mathrm{Spec}(\mathbb{C}[t])$$
$$x \longmapsto x^2$$

Not étale 2:1 covering except for 0

$$\mathrm{Spec}(\mathbb{C}[t, t^{-1}]) \longrightarrow \mathrm{Spec}(\mathbb{C}[t])$$
$$x \longrightarrow x^2$$

is étale.

étale : covering map + open immersion

Étale presheafs

$$\mathcal{F} : \mathrm{Top}_{\acute{e}t}(X)^{opp} \longrightarrow \Bbbk\text{-mod}.$$

$\mathrm{Sh}_{\acute{e}t}(X)$, $D^{+}_{\acute{e}t}(X;\Bbbk)$, $\mathrm{Perv}(X;\Bbbk)$, $f_!$, f_*, ... etc

e.g. Tate modules $m \in \mathbb{Z}$

$$U \longrightarrow \operatorname{Spec} \overset{X}{\overset{\shortparallel}{\mathbb{F}_q}} \qquad U \text{ connected} \Rightarrow U = \operatorname{Spec} \mathbb{F}'$$

$\mathbb{F}'/\mathbb{F}_q$ finite field extension.

$m \in \mathbb{Z}_{\geq 0}$.

$$\underline{\mathbb{Z}/m\mathbb{Z}}_X(1)(\operatorname{Spec}\mathbb{F}' \longrightarrow \operatorname{Spec}\mathbb{F}_q) := \{ b \in \mathbb{F}' : b^m = 1 \} =: \mu_m$$

$$\in \mathbb{Z}/m\mathbb{Z}\text{-mod}.$$

Show this is sheaf.

More general, for $X / \mathbb{F}_q$ variety, the Tate sheaf is the sheaf given by

$$\underline{\mathbb{Z}/m\mathbb{Z}}_X(1)(U \to X) := \{ f \in \mathbb{F}_q[U] \mid f^m = 1 \}$$

locally constant, locally $\underline{\mathbb{Z}/m\mathbb{Z}}_X$

stalks $= \mathbb{Z}/m\mathbb{Z}$

$\mathbb{F}_q \not\supseteq \mu_m$ this is not the constant sheaf $\underline{\mathbb{Z}/m\mathbb{Z}}_X$

<u>e.g.</u> $X = \operatorname{Spec}(\mathbb{F}_2)$ Peter : $\pi_1^{ét} \neq 1$.

x^3-1 separable $|\mu_3| = 3$ $\# \mathbb{F}_2 = 2$. $\mu_3 \notin \mathbb{F}_2$

$\rightsquigarrow$ $\underline{\mathbb{Z}/3\mathbb{Z}}_X(1)$ not constant

$Sh_{ét}(*)$ is not k-mod!.

$Sh(*) = k\text{-mod}$ $\operatorname{Spec}(k)$ $B\,\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$

Étale sheaves over $\mathbb{Z}$ or $\mathbb{Q}$ are not considered.

Tate sheaf over $\mathbb{Z}_\ell$:

$$\underline{\mathbb{Z}_\ell}_X(1) := \varprojlim \underline{\mathbb{Z}/\ell^i\mathbb{Z}}_X(1)$$

Arithmetic Frobenius:

$$\begin{aligned} \mathrm{Fr}_q^{\mathrm{arith}} : \overline{\mathbb{F}_q} &\longrightarrow \overline{\mathbb{F}_q} \\ x &\longmapsto x^q \end{aligned}$$

Geometric Frobenius:

$$\mathrm{Fr}_q := \left(\mathrm{Fr}^{\mathrm{arith}}\right)^{-1}$$

$\mathrm{Fr}_q^{\mathrm{arith}}, \mathrm{Fr}_q \in \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ are topological generators

A continuous action of $G = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ is determined by Fr.

char $\mathbb{F}_q$ fun

e.g. $X = \operatorname{Spec} \mathbb{F}_q$. $\bar{x} : \operatorname{Spec}(\overline{\mathbb{F}_q}) \longrightarrow \operatorname{Spec}(\mathbb{F}_q)$

$G = \operatorname{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ $\curvearrowright$ $\{ \zeta \in \overline{\mathbb{F}_q} \mid \zeta^m = 1 \}$ Choice of

$\|$ $1 \in \mathbb{Z}/m\mathbb{Z}$

$$\mathrm{Fr}_q^{\text{arith}} : \underline{\mathbb{Z}/m\mathbb{Z}}(1)_{\bar{x}} \longrightarrow \underline{\mathbb{Z}/m\mathbb{Z}}(1)_{\bar{x}}$$

$\mathbb{Z}/m\mathbb{Z}$-module. $\zeta \longmapsto \zeta^q$ multiplication by q in $\mathbb{Z}/m\mathbb{Z}$

Cyclotomic character.

let $m = \ell^i$ ℓ prime $\neq p$. We set

$\mathrm{Fr}_q^{\text{arith}} : \underline{\mathbb{Z}_\ell}(1)_{\bar{x}} \longrightarrow \underline{\mathbb{Z}_\ell}(1)_{\bar{x}}$ is given by q

$\mathrm{Fr}_q : \underline{\mathbb{Z}_\ell}(1)_{\bar{x}} \longrightarrow \underline{\mathbb{Z}_\ell}(1)_{\bar{x}}$ is given by q^{-1}

$\overline{\mathbb{Q}_\ell}$ $F/\mathbb{Q}_\ell$.

§2. The pure cohomology.

Complex absolute value

$$\begin{tikzcd} \mathbb{C} \arrow[r, dash, "\iota"] \arrow[d, dash] & \mathbb{C}_\ell \arrow[d, dash] \\ \overline{\mathbb{Q}_\ell} \arrow[r, equal, "\mathrm{id}"'] & \overline{\mathbb{Q}_\ell} \end{tikzcd}$$

$$\begin{tikzcd} & \mathbb{R} = \mathbb{Q}_\infty \arrow[r, "\text{alg. cl}"] & \mathbb{C} \arrow[d, dash, "\simeq"] \arrow[dr, dash, "\simeq"] & \\ \mathbb{Q} \arrow[ur, "|\cdot|"] \arrow[dr, "|\cdot|_\ell"'] & & & \mathbb{C}_\ell \\ & \mathbb{Q}_\ell \arrow[r, "\text{alg. cl}"] & \overline{\mathbb{Q}_\ell} \arrow[ur, "\text{Top completion}"'] & \end{tikzcd}$$

complex field of ℓ-adic numbers

$F/\mathbb{Q}_\ell$ $\quad$ $\mathbb{Q}_\ell(\alpha)$

$\alpha \in \overline{\mathbb{Q}_\ell}$ has complex absolute value

$\iota : \overline{\mathbb{Q}_\ell} \longrightarrow \mathbb{C}$ $\quad$ $|\iota(\alpha)| = a$.

$\mathbb{Q}$, μ_n, In practice, $\overline{\mathbb{Q}} \ni \alpha$ $\quad$ $\frac{1+\sqrt{5}}{2} \in \overline{\mathbb{Q}}$ $\quad$ $\frac{1-\sqrt{5}}{2}$ ✗.

$|\pm i| = 1$.

Let V a $\mathbb{Z}$-graded $\mathbb{Q}_\ell$-vector space

$$V = \bigoplus_{i \in \mathbb{Z}} V_i \qquad \dim V_i < \infty .$$

$$F \in \mathrm{End}^{\cdot}_{\mathbb{Q}_\ell}(V)$$

$\mathbb{Z}_{>0}$
$\Downarrow$

F is of weight $\leq w$ (resp. pure weight w) w.r.t. q

if e.v. of F in V_i in $\overline{\mathbb{Q}_\ell}$ have complex absolute value equal to $q^{j/2}$ $\quad j \leq w + i$

(resp $j = w + i$)

let X_0 proper variety over $\mathbb{F}_q$ (denote by X for it over $\overline{\mathbb{F}_q}$)

$$Fr \curvearrowright H^*_{ét}(X, \mathbb{Q}_\ell)$$

Deligne's theorem $\rightarrow$ $Fr \curvearrowright H^*$ is of weight ≤ 0.

We define the pure part of $H^*(X,\mathbb{Q}_\ell)$ by quotient out negative weight generalised eigenspaces.

$$H^*_p(X;\mathbb{Q}_\ell)$$

let $j : U_0 \hookrightarrow X_0$ smooth locus of X_0

$$\underline{\mathbb{Q}_\ell}_X \longrightarrow j_{!*}(\underline{\mathbb{Q}_\ell}_U) =: IC(X;\mathbb{Q}_\ell)$$

$$\mathcal{H}^i(\ ) = \begin{cases} 0 & i<0 \\ \underline{\mathbb{Q}_\ell} & i=0 \\ * & i>0 \end{cases}$$

${}^0j_*\underline{\mathbb{C}}_U = \underline{\mathbb{C}}_X \iff$ unibranched

we get a map

$$\mathbb{H}^*(X;\underline{\mathbb{Q}_\ell}_X) \longrightarrow \mathbb{H}^*(X; j_{!*}\underline{\mathbb{Q}_\ell}) = IH^*(X;\underline{\mathbb{Q}_\ell})$$

BE 2009 $\longrightarrow$ ker = e.s. of wt < 0. Then

$$\mathbb{H}^*_p(X;\underline{\mathbb{Q}_\ell}_X) \hookrightarrow IH^*(X;\mathbb{Q}_X).$$

"proof":

$$\underset{\text{wt} < 0}{\mathcal{F}} \to \underline{\mathbb{Q}_\ell}_X \longrightarrow \underset{\text{pure wt} = 0.}{j_{!*}\underline{\mathbb{Q}_\ell}_U} \longrightarrow \cdots$$

Betti numbers

$$b_i := \dim_{\mathbb{Q}_\ell} H^i(X;\mathbb{Q}_\ell)$$

Pure Betti numbers

$$b_i^p := \dim_{\mathbb{Q}_\ell} H^i_p(X;\mathbb{Q}_\ell)$$

§3. Hard Lefschetz

Let X_0 be a projective variety over $\mathbb{F}_q$, of pure dimension n.

<u>Thm</u>.

$$b_i^p \le b_{i+2j}^p \qquad 0 \le j \le n-i \qquad (*)$$

$$b_{n-i}^p \le b_{n+i}^p \qquad i \le n$$

<u>Proof</u> $j : U_0 \hookrightarrow X_0$

$\underline{\mathbb{Q}_\ell}_X \longrightarrow j_{!*}\underline{\mathbb{Q}_\ell}_U$ map of $\underline{\mathbb{Q}_\ell}_X$-modules

$\rightsquigarrow \mathbb{H}^*(X;\underline{\mathbb{Q}_\ell}_X) \longrightarrow I\mathbb{H}^*(X,\underline{\mathbb{Q}_\ell}_X)$

$\mathbb{H}^*(X,\underline{\mathbb{Q}_\ell}_X)$-map.

Hyperplane $\rightsquigarrow$ line bundle $\longrightarrow$ characteristic of line bundle

$H \rightsquigarrow \mathcal{L}_H \rightsquigarrow c_1(\mathcal{L}) \in H_p^2(X;\mathbb{Q}_\ell)$

first Chern class

$$\begin{tikzcd} H_p^i(X;\mathbb{Q}_\ell) \arrow[r, hook] \arrow[d, "\cap(c_1(\mathcal{L}_D))^j"'] & IH^i(X;\mathbb{Q}_\ell) \arrow[d, "\cap(c_1(\mathcal{L}_D))^j", "\cong"'] \\ H_p^{i+2j}(X;\mathbb{Q}_\ell) \arrow[r, hook] & IH^i(X;\mathbb{Q}_\ell) \end{tikzcd}$$

Hard Lefschetz theorem

$\Rightarrow b_p^i \leq b_p^{i+2j}$ $\square$.

$j = n-i$ HL

$b^\rho_{n-i} \leq b^\rho_{n+i}$

§4. Betti numbers

Number of cells.

Let X a stratified proper variety. $\{C_\alpha\}_\alpha$

$\alpha \leq \beta$ if $C_\alpha \subseteq \overline{C_\beta}$

$$X = \bigsqcup_\alpha C_\alpha$$

$$X_\beta = \bigsqcup_{\alpha \leq \beta} C_\alpha = \overline{C_\beta}$$

An algebraic cell decomposition of X is a stratification s.t

$\forall \alpha \quad C_\alpha \cong \mathbb{A}^n$ for some n.

Thm. Let $f_i :=$ number of cells of $\dim = i$.

(i) $H^{2i+1}(X; \mathbb{Q}_\ell) = 0 \quad \forall i$. In particular

$$b_{2i+1} = b^p_{2i+1} = 0$$

(ii) $H^{2i}(X; \mathbb{Q}_\ell) = H^{2i}_p(X; \mathbb{Q}_\ell) \quad \forall i$. In particular

$$b_{2i} = b^p_{2i}$$

(iii) If X is projective of pure dimension n

$$f_i \leq f_j \quad \text{for all } i \leq j \leq n-i$$

Proof:

$$H^i_c(\mathbb{A}^n) = \begin{cases} 0 & i = 0 \\ 0 & i \neq 2n \\ \mathbb{Q}_\ell & i = 2n \end{cases}$$

compactly supported cohomology

$U \subseteq X$ $\qquad U \subset X \supset X \setminus U$

$$H^i_c(U;\mathbb{Q}_\ell) \to H^i_c(X;\mathbb{Q}) \to H^i_c(X\setminus U;\mathbb{Q})$$

$$\qquad\qquad\qquad\qquad \| \qquad\qquad\qquad\qquad \|$$

$$H^i(X;\mathbb{Q}) \to H^i(X\setminus U;\mathbb{Q})$$

$$(X \setminus U_1) \setminus U_2 \quad \cdots \quad \Rightarrow \quad i) \ \& \ ii)$$

iii) $\quad f_i \leq f_j \quad$ for all $\quad i \leq j \leq n-i \qquad (*)$

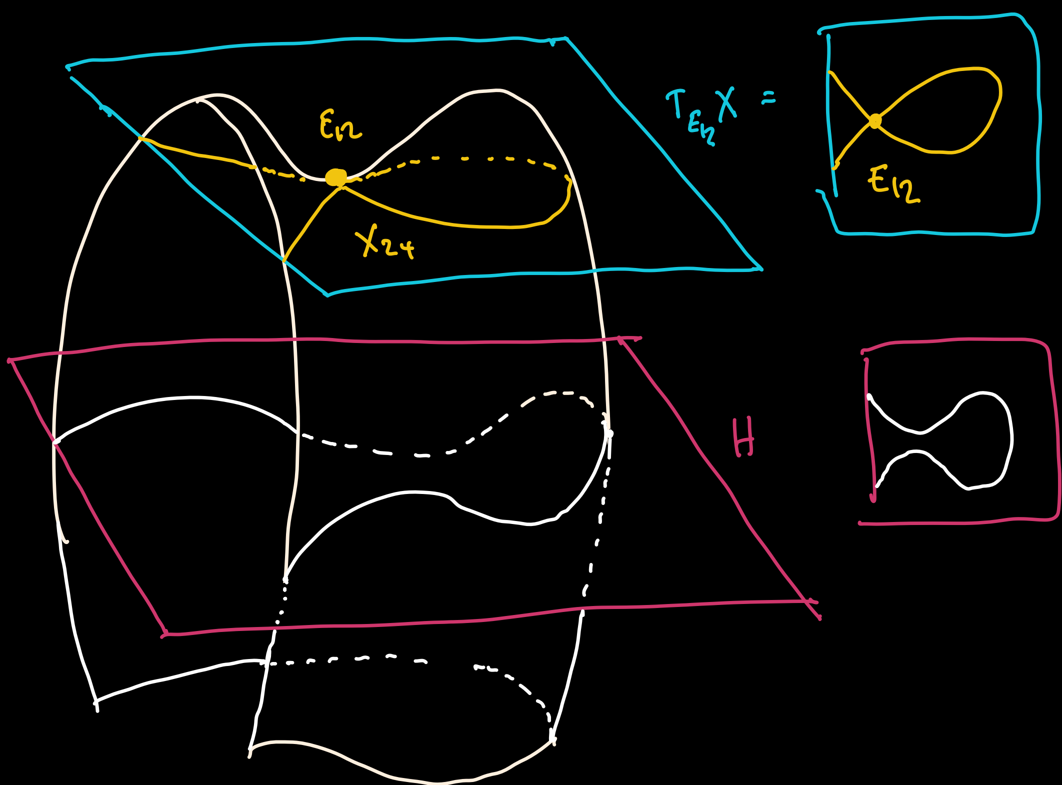
e.g. $X_{24} \subseteq X = Gr(2,4)$.

$E_{12} = \langle e_1, e_2 \rangle \subseteq \mathbb{C}^4$

$$X_{24} = \{ E \subset \mathbb{C}^4 \mid \dim E = 2,\ \dim(E \cap E_{12}) \geq 1 \}$$

How to compute $IH^\bullet(X_{24})$

$X_{24} = X \cap T_{E_{12}} X$



$$X_{24} = \mathrm{Proj}\left(\mathbb{C}[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}] \big/ \langle p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23},\ p_{34} \rangle \right)$$

$$E_{12} = \langle e_1, e_2 \rangle \subseteq \mathbb{C}^4$$

$$X_{24} = \{ E \subset \mathbb{C}^4 \mid \dim E = 2,\ \dim(E \cap E_{12}) \geq 1 \}$$

$$\tilde{X}_{24} = \left\{ (l, E) : \begin{array}{l} \dim l = 1 \\ \dim E = 2 \end{array},\ l \subset E \cap E_{12} \right\}$$

$$\begin{aligned} \tilde{X}_{24} &\longrightarrow X_{24} \\ (l, E) &\longmapsto E \end{aligned}$$

$$\begin{tikzcd} & \tilde{X}_{24} \ni (l,E) \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\ l' \in \mathbb{P}^1 & & X_{24} \ni E \end{tikzcd}$$

- $\tilde{X}_{24}$ is smooth

$$\begin{aligned} p : \tilde{X}_{24} &\longrightarrow \mathbb{P}^1 \\ (l, E) &\longmapsto l_{12} \end{aligned}$$

$$\begin{array}{c} \mathbb{C}^2 \\ \| \\ l \subseteq E_{12} \subseteq \mathbb{C}^4 \\ \rotatebox{90}{$\leftrightsquigarrow$} \\ l_{12} \subseteq \mathbb{C}^2 \\ l_{12} \in \mathbb{P}^1 \end{array}$$

what is the fibre of l_{12}?

$$\begin{array}{c} l_{12} \subseteq \mathbb{C}^2 \\ \downarrow \\ l'_{12} \subseteq E_{12} \end{array}$$

X_{34}

$$p^{-1}(l_{12}) = \{ (l'_{12}, E) : l'_{12} \subseteq E \subseteq \mathbb{C}^4 \}$$
$$\simeq \{ 0 \subseteq e \subseteq \mathbb{C}^3 \mid \dim e = 1 \} \simeq \mathbb{P}^2$$

$\tilde{X}_{24} \longrightarrow \mathbb{P}^1$ is a $\mathbb{P}^2$-fibration over $\mathbb{P}^1 \Rightarrow \tilde{X}_{24}$ smooth.

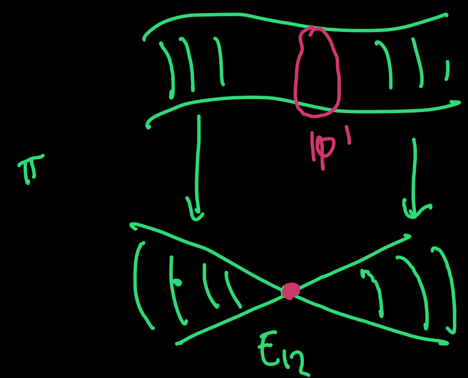
- $\tilde{X}_{24} \xrightarrow{\pi} X_{24}$ is small resolution

  $(\ell, E) \longmapsto E$

$X_{24} = \{E \subset \mathbb{C}^4 \mid \dim E = 2,\ \dim(E \cap E_{12}) \geq 1\}$

$\pi^{-1}(E_{12}) = \{\ell \subset E_{12}\} = \mathbb{P}^1$

$E \neq E_{12}$, $\pi^{-1}(E) = \{\ell \subseteq E \cap E_{12} (= \ell)\} = \{E \cap E_{12}\} = *$



$Gr(2,4) \subset \mathbb{P}^5$

$\{U_{12} \neq 0\}$ $X_{34} \cap U_{12} \subset \mathbb{A}^4$

$X_{24} \cap U_{12} = C_{24} \cup \{E_{12}\} \subset \mathbb{A}^4$

$\frac{\dim C_{24}}{2} = \frac{3}{2} > \dim \mathbb{P}^1 = 1$

small resolution!

- $\pi : \tilde{X} \to X$ small

$$IH^\bullet(X) = H^\bullet(\tilde{X})$$

- Since $\tilde{X}_{24}$ is $\mathbb{P}^2$-bundle over $\mathbb{P}^1$

$$H^\bullet(\tilde{X}_{24}) = H^\bullet(\mathbb{P}^1 \times \mathbb{P}^2, \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0 \\ \mathbb{Q} \oplus \mathbb{Q} & i = 2 \\ \mathbb{Q} \oplus \mathbb{Q} & i = 4 \\ \mathbb{Q} & i = 6 \end{cases}$$

Künneth formula

$$H^i(X \times Y ; \mathbb{Q}) = \bigoplus_{n+m=i} H^n(X) \otimes H^m(Y)$$

$$\begin{tikzcd}
 & H^*(X_{24}) & IH^*(X_{24}) & \\
6 & \mathbb{Q} \arrow[r, hook] & \mathbb{Q} & \\
4 & \mathbb{Q} \oplus \mathbb{Q} \arrow[r, hook] & \mathbb{Q} \oplus \mathbb{Q} & \\
c_1(\mathcal{L}) & \arrow[u, hook, "HL"'] & \arrow[u, hook, "HL"] & \\
2 & \mathbb{Q} \oplus \mathbb{Q} \arrow[r, hook] & \mathbb{Q} \oplus \mathbb{Q} & \\
0 & \underbrace{\mathbb{Q}} \arrow[r, hook] & \mathbb{Q} &
\end{tikzcd}$$

$$b_2^p \leq b_4^p$$
$$\begin{matrix} \| & & \| \\ 2 & \leq & 2 \end{matrix}$$

$1 = b_2 \leq b_4 = 2.$

$$\mathbb{Q} \longrightarrow \mathbb{Q} \hookrightarrow \mathbb{Q} \oplus \mathbb{Q} \longrightarrow ?\, \mathbb{Q}$$

HL completion $\qquad c_1(\underbrace{\mathcal{O}_X(1)})$

$E_{12} \to \langle e_1, e_2 \rangle \qquad$ 2-bundle E

$c_1(E) \quad c_2(E)$

$$
\begin{array}{cccc}
 & H^*(X_{24}) & & IH^*(X_{24}) \\
6 & \mathbb{Q} & \hookrightarrow & \mathbb{Q} \\
4 & \mathbb{Q}\oplus\mathbb{Q} & \hookrightarrow & \mathbb{Q}\oplus\mathbb{Q} \\
2 & \mathbb{Q} & \hookrightarrow & \mathbb{Q}\oplus\mathbb{Q} \\
0 & \mathbb{Q} & \hookrightarrow & \mathbb{Q}
\end{array}
$$

§4. The shape of Bruhat intervals

Coxeter groups ?

$\cup$

Crystallographic Coxeter groups $\rightsquigarrow$ Kac-Moody Lie algebras

$\cup$ Affine Weyl groups

$\cup$ Weyl groups

$\cup$ Affine Lie algebras

$\cup$ s.s. Lie algebras

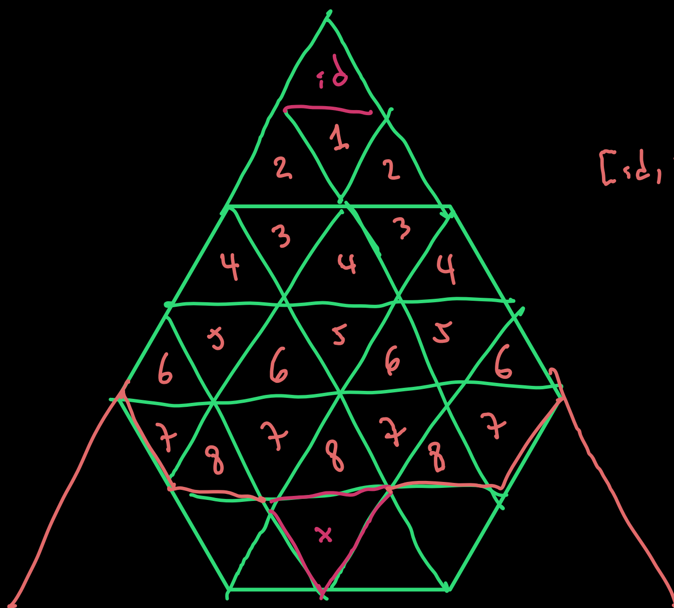
Geometry of flag varieties

(W,S) Weyl group. $J \subseteq S$, $W_J \leq W$

$$W^J = W_J \backslash W$$

$f_i^{J,w}$ = # of i dimensional cells in G/P associated to W^J

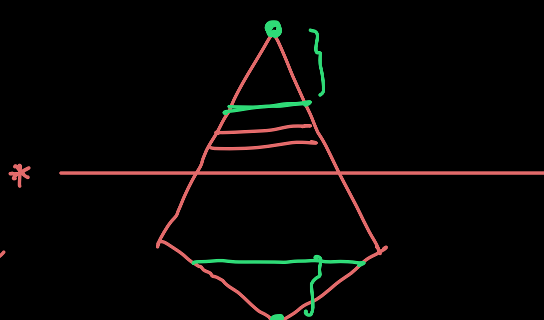
$$f_i^{J,w} \leq f_j^{J,w} \quad \text{for} \quad i \leq j \leq n-i$$



$[id, x]$

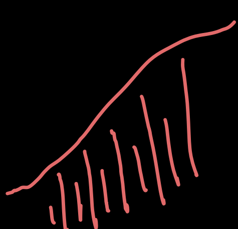
$b_0 = 1 \leq b_9 = 1$ $\qquad$ $b_1 = 1 \leq b_8 = 3$

$b_1 \leq b_8$ $\qquad$ $2 = b_2 \leq b_7 = 4$



$J \subset S$ $\qquad$ $W_J \backslash W$

$S - \{s_0\}$



$rk = 2$

$\underline{Gr(4,12)}$