The Bruhat graph of a Coxeter group

References:

S.A. M. Dyer 1990 "Reflection subgroups of Coxeter systems"

Comp. " 1991 "On the "Bruhat graph" of " " "

Comp. " 1993 "Hecke algebras and shellings of Bruhat intervals"

Plan:

① Motivation

② Basic results

③ Proof of Miracle 3.

§1. Motivation

Let (W,S) be a finite Coxeter system

$T := \bigcup_{w \in W} wSw^{-1} \xleftrightarrow{\sim} \left\{ \begin{matrix} \text{positive} \\ \text{roots} \end{matrix} \right\}$

set of reflections

Let $\ell : W \to \mathbb{N}$ be the length function

Def The Bruhat graph $\Omega_{(W,S)}$ is the directed graph with.

Vertex set : W

Edge set : $E_{(W,S)} := \left\{ (tw, w) \;\middle|\; \begin{matrix} w \in W \\ t \in N(w) \end{matrix} \right\}$

where

$N(w) := \{ t \in T \mid \ell(tw) < \ell(w) \}$

If $(x,y) \in E_{(W,S)}$ we sometimes will label the edge with $xy^{-1} \in T$.

⊛ Some questions about Coxeter groups can be solved only in terms of the Bruhat order.

Def A path Δ of $\Omega_{(W,S)}$ from x to y is a sequence $(x_0 = x, x_1, x_2, \dots, x_n = y) \in W^n$ s.t. $(x_i, x_{i+1}) \in E_{(W,S)}$.

The Bruhat order $\leq$ of W is defined by $x \leq y$ iff $\exists$ path from x to y in Ω.

If $X \subset W$ denote by $\Omega_{(W,S)}(X)$ to be the full subgraph of $\Omega_{(W,S)}$ containing X as vertex set.

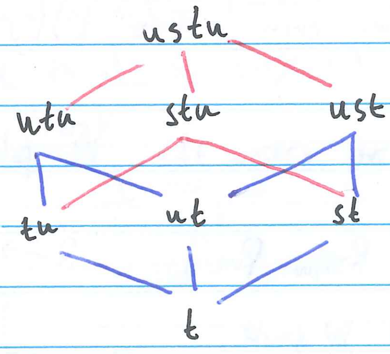
Miracle 1 One can determine whether $xy^{-1} \in T$ using only $\leq$.

e.g. $S_4 = \langle s,t,u \mid sts = tst,\ utu = tut,\ su = us \rangle$ $\quad s^2 = t^2 = u^2 = e$

Clearly $utstu \in T$.

Does $tustu \in T$? (i.e. $(t, ustu) \in E_{(W,S)}$)

Let's see the poset structure of $[t, ustu]$



3-crown!

By Dyer 91 (Miracle 2) a 3-crown cannot produce the relation $xy^{-1} \in T$. interval $[x,y]$

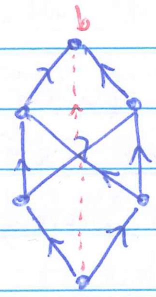
Miracle 2

Prop (Dy 91) $I \subset W$ interval. Can construct $\Omega_{(W,S)}(I)$ using only $(I, \leq)$ in the following way.

① Let $E_1 = \{ (x,y) \in I \times I \mid x \lessdot y \}$.

Let $B \supset E_1$ the minimal set containing E s.t.

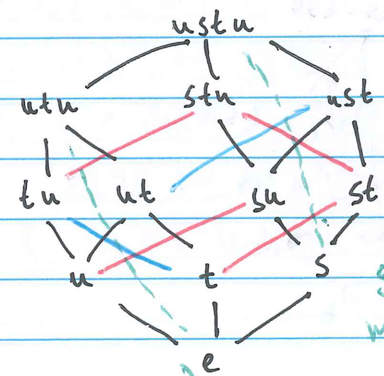
If



$\in B$ then $(a,b) \in B$.

Then $B = \Omega_{(W,S)}(I)$

e.g. S_4 Bruhat subgraph. $\Omega_{(W,S)}([e, ustu])$



Therefore $utu \in T$. $sustu \in T$.

green dotted lines were added using $(ab \in T \Leftrightarrow ba \in T)$

Let (W,S) be any Coxeter system.

<u>Conjecture</u> (Combinatorial invariance conjecture)
Kazhdan–Lusztig 1979

If $[x,y] \cong [\tilde{x},\tilde{y}]$ then

$$P_{x,y} = P_{\tilde{x},\tilde{y}}.$$

(<u>Rmk</u> Open problem even in finite type (e.g. type A))

R-polynomials are easier than KL-polys.

$$q^{\ell(x,y)} P_{x,y} = \sum R_{x,w} R_{w,y} \qquad \tilde{R} \leftrightarrow R.$$

<u>Thm</u> (Dyer 1993) W finite.

$$\tilde{R}_{x,y} = \sum_{\substack{\Delta \text{ "nice" paths} \\ \text{from } x \text{ to } y \\ \text{in } \Omega([x,y])}} q^{|\Delta|}$$

("nice" means descendent set is ϕ for a choice of a total order in Φ^+)

This theorem and Miracle 2 give support to the conjecture. But the description still involves T (or Φ^+).

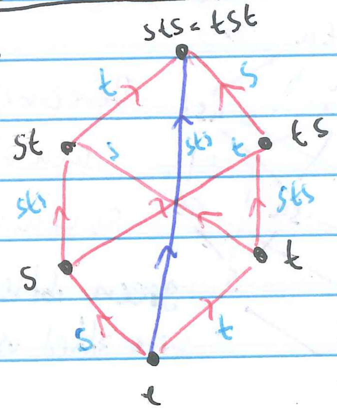
<u>Q.</u> How many iso types of Bruhat intervals are possible?

<u>Miracle 3</u>

Let $n \in \mathbb{N}$ fixed. There are finitely many iso types (as posets) of intervals $[x,y]$ of length n. (i.e. $\ell(y)-\ell(x)=n$) occurring in finite ~~class~~ type Coxeter systems.

§2. <u>Basic results</u>

$A_2 = D_6$

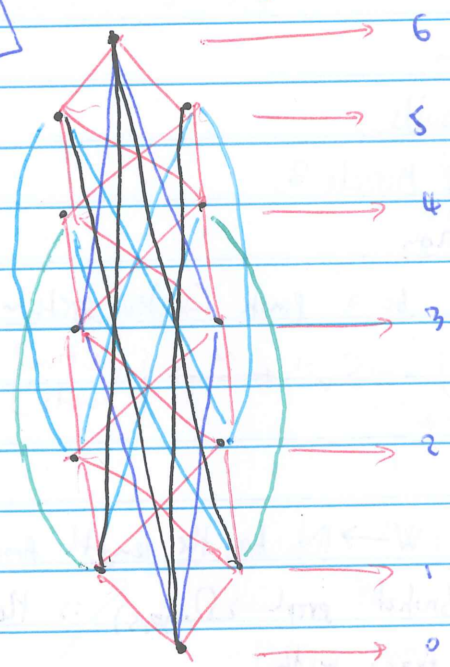


2) W dihedral st. $I = [x,y] \subset W$
s.t. $\ell(y) - \ell(x) = n$.

Let $Z \xrightarrow{f} \{0,1,\dots,n\}$

st $\#f^{-1}(i) = \begin{cases} 1 & \text{if } i=1,n \\ 2 & \text{o/w.} \end{cases}$

$\boxed{D_{12}}$



Z

$(x,y) \in E(\Omega(I))$ if
$f(y) - f(x)$ odd
and $f(x) < f(y)$

<u>Def</u> (Dyer 90) A subgroup W' of W is a reflection subgroup if
$W' = \langle W' \cap T \rangle$.

<u>Thm</u> (Dy 90) Let $W' \subset W$ be a ref subgroup
$S' = \{t \in T \mid N(t) \cap W' = \{t\}\} =: \chi(W')$
then (W',S') is a Coxeter system.

<u>Prop</u> (Dy 90)
$W' \subset W$ ref. sub. $S' = \chi(W')$.

(i) $\Omega_{(W',S')} = \Omega_{(W,S)}(W')$

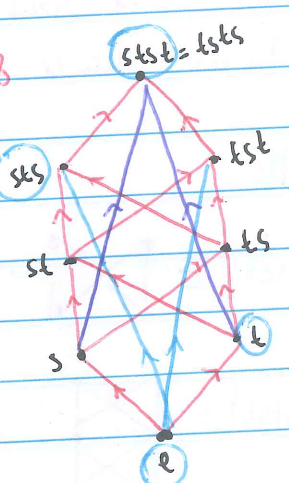
(ii) For $x \in W$, let x_0 be the minimal length representative of the right coset $W'x$. Then the map $w \longmapsto wx_0$ induces a directed graph isomorphism:

$$\Omega_{(W',S')} \xrightarrow{\sim} \Omega_{(W,S)}^{(W'x)}$$

Example

$W = D_8$

$\Omega_{(W,S)} =$



$W' = \langle sts, t \rangle = \{e, t, sts, stst\}$

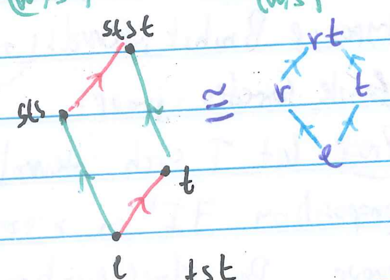
$x = ts$

$W'x = W'ts = \{ts, s, tst, st\}$

$x_0 = s \rightarrow$ minimal length representative of $W'ts$

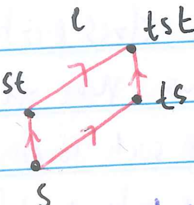
The proposition says the map $w \longmapsto ws$ induces iso of directed graphs between $\Omega_{(W',S')}$ and $\Omega_{(W,S)}^{(W'ts)}$

$\Omega_{(W',S')} =$



$\updownarrow s\ (\cdot s)$

$\Omega_{(W,S)}^{(W'ts)} =$



The map does not preserve difference of lengths!

§3. Proof of Miracle 3

Prop (Dy §1)

Let (W,S) be a finite Coxeter group $I = [x,y]$, $n := \ell(y) - \ell(x)$. Then $W' := \{uv^{-1} \mid u, v \in I\}$ is a reflection subgroup of rank $\leq n$. (i.e. $\# \chi(W') \leq n$). Moreover, there exist an interval I' in W' such that

$$I \cong I'$$

poset isomorphism.

Proof:

$n=0$, $I = \{x\}$, $W' = \{e\} = I'$

$\chi(W') = \phi$

$n=1$, $I = [tw, w] = \{tw, w\}$

$W' = \{e, t\} = [e,t] = I' \cong I$.

$n \geq 2$

Let $(x_0, x_1, \dots, x_n)$ be a path in $\Omega_{(W,S)}$ from x to y (i.e. $x_0 = x$, $x_n = y$)

Define $W'' = \langle \{x_{i-1} x_i^{-1}\} \mid i \in \{1, \dots, n\} \rangle$

So W'' is a reflection subgroup.

Consider $W''x$ and let z be the minimal length representative of $W''x$.

By the proposition before the map $w \longmapsto wz$ induces an iso of directed graphs

$$\Omega_{(W'',S'')} \xrightarrow{\sim} \Omega(W''x)$$

Let $(x_0 z^{-1}, x_1 z^{-1}, \dots, x_n z^{-1})$ is a path from xz^{-1} to yz^{-1} in $\Omega_{(W'',S'')}$ where $S'' = \chi(W'')$. Then,

$$\ell''(yz^{-1}) - \ell''(xz^{-1}) \geq n$$

where ℓ'' is the length of W''.

Let $(w_0, \dots, w_m)$ any path from xz^{-1} to yz^{-1} in W'', then $(w_i z)_{i=0}^{m}$ is a path from x to y.

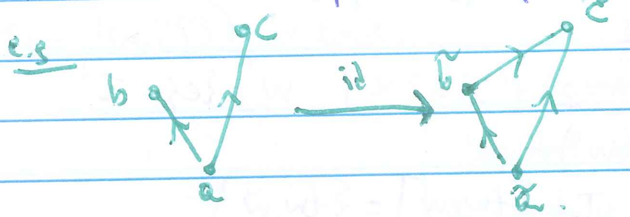
$\therefore \ \ell''(yz^{-1}) - \ell''(xz^{-1}) \le n$.

Define $I' = [xz^{-1}, yz^{-1}] \subset W''$, and

$$I' \xrightarrow{f} I$$
$$w \longmapsto wz.$$

this map must be injective and order preserving. But is still not clear that is an isomorphism of posets.

e.g



the map "id" is order preserving bijection but the target set has the extra relation $\tilde b < \tilde c$. However, b and c are not comparable.

By a theorem of Bjorner and Wachs, 1982 if $[x,y]$ is a length "$n \ge 3$" interval then $[x,y] \setminus \{x,y\}$, viewed as an abstract simplicial complex is a combinatorial $n-2$ sphere.

If $n=2$, our f is a poset isomorphism. Suppose $n \ge 3$. f maps injectively

$$\underbrace{[xz^{-1}, yz^{-1}] \setminus \{xz^{-1}, yz^{-1}\}}_{S^{n-2} \text{ comb. sphere}} \text{ in } \underbrace{[x,y] \setminus \{x,y\}}_{S^{n-2} \text{ comb sph.}}$$

the only way to have an inclusion between two $(n-2)$-homogeneous boundaryless connected manifolds is via an isomorphism.

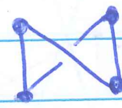
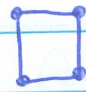
Therefore $f: I' \to I$ is a poset isomorphism.

By a theorem of Dyer, the rank of a refl. subgroup is bounded by the number of generators, then

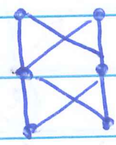
$\# \chi(W'') \le n$.

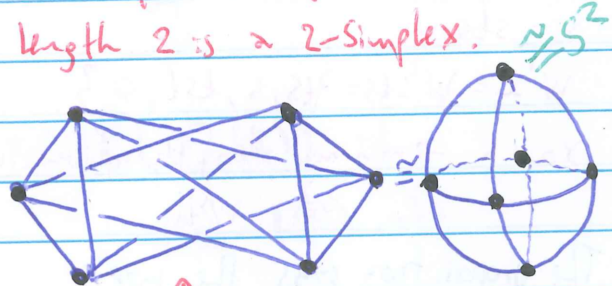
Now $W'' \subset W'$. Also $W' \subseteq W''$ since for $u, v \in I'$ we have $f(u) f(v)^{-1} = uv^{-1}$. Hence $W' = W''$ □.

e.g $W = S_3 = D_6$,

$[e, sts] \setminus \{e, sts\} =$  $\sim$  $\cong S^1$

$W = D_8$

$[e, stst] \setminus \{e, stst\} =$  As an abstract simplicial complex every chain of length 2 is a 2-simplex.



8 triangles

Corollary (Miracle 3) For each $n \in \mathbb{N}$, there are only finitely many isomorphism types of Bruhat intervals of length n occurring in finite Coxeter groups.

Proof Let I such interval. By previous proposition $\exists I'$ in a rank n finite Coxeter group. By classification of finite Cox. systems there are finitely many families of such Coxeter systems, each one contains only finitely many iso types of Bruhat intervals □. THE END.