The Column Space and the Null Space of a Matrix

- Suppose that A is a $m \times n$ matrix. Then

$$\dim \mathrm{Null}(A) + \dim \mathrm{Col}(A) = n.$$

  Why:

  - $\dim \mathrm{Null}(A) =$ number of free variables in row reduced form of A.
  - a basis for $\mathrm{Col}(A)$ is given by the columns corresponding to the leading 1's in the row reduced form of A.

- The dimension of the Null Space of a matrix is called the "nullity" of the matrix.

- The dimension of the Column Space of a matrix is called the "rank" of the matrix.

Linear transformations and matrices

Suppose that V and W are two vector spaces.
A linear transformation is a function $f : V \longrightarrow W$ such that

$$f(rx + sy) = r f(x) + s f(y),$$

for all $x, y \in V$ and $r, s \in \mathbb{R}$.

Let A be an $n \times m$ matrix.
Then A determines a linear transformation $f_A : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ which is given by:
$f_A(x) = Ax$, for $x \in \mathbb{R}^m$.

To check this is is enough to observe that if $x, y \in \mathbb{R}^m$ and $r, s \in \mathbb{R}$ then

$$f_A(rx+sy) = A(rx+sy) = rAx+sAy = rf_A(x)+sf_A(y),$$

by the usual rules for matrix multiplication.

Hence, f_A is a linear transformation as claimed.

Linear transformations: $\mathbb{R} \to \mathbb{R}$

Question :
When is a map $f : \mathbb{R} \longrightarrow \mathbb{R}$ a linear transformation?

Write $f(1) = a$, for some $a \in \mathbb{R}$.

Then $f(2) = f(1+1) = f(1)+f(1) = a+a = 2a$

Alternatively, $f(2) = f(2 \cdot 1) = 2f(1) = 2a$.

More generally, $f(u) = uf(1) = ua$, for any $r \in \mathbb{R}$.

Thus, the linear transformations from $\mathbb{R}$ to $\mathbb{R}$ are the maps $f(u) = ua$, for some $a \in \mathbb{R}$.

Question : What does this correspond to geometrically?

Linear transformations: $\mathbb{R} \to \mathbb{R}^2$

Question :
When is a map $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ a linear transformation?

As $\mathbb{R}^2 = \{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in \mathbb{R} \}$ if $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ is a map then we can write $f(1) = \begin{bmatrix} a \\ b \end{bmatrix}$, for some $a, b \in \mathbb{R}$.

As f is a linear transformation

$$f(u) = f(u \cdot 1) = uf(1) = u \begin{bmatrix} a \\ b \end{bmatrix}$$
$$= \begin{bmatrix} ua \\ bu \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} u \end{bmatrix}.$$

Notice that the last equality expresses f in terms of matrix multiplication.

Linear transformations $\mathbb{R}^2 \to \mathbb{R}$

Question :
When is a map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ a linear transformation?

Recall that $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^2$.

Write $f \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a$ and $f \begin{bmatrix} 0 \\ 1 \end{bmatrix} = c$

Then, $f \begin{bmatrix} u \\ v \end{bmatrix} = uf \begin{bmatrix} 1 \\ 0 \end{bmatrix} + vf \begin{bmatrix} 0 \\ 1 \end{bmatrix} = ua + vc = (ua + vc)1$
$= \begin{bmatrix} a & c \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$.

Again, the last equality expresses f in terms of matrix multiplication.

Linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$

Question :
What are the linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$?

As before, write $f\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $f\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$, for some $a, c \in \mathbb{R}$.

$$\text{Then } f\begin{bmatrix} u \\ v \end{bmatrix} = uf\begin{bmatrix} 1 \\ 0 \end{bmatrix} + vf\begin{bmatrix} 0 \\ 1 \end{bmatrix} = u\begin{bmatrix} a \\ b \end{bmatrix} + v\begin{bmatrix} c \\ d \end{bmatrix}$$
$$= \begin{bmatrix} ua \\ ub \end{bmatrix} + \begin{bmatrix} vc \\ vd \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}.$$

Once again, the last equality expresses f in terms of matrix multiplication.

This shows that every linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is given by left multiplication by a 2×2 matrix.

Linear transformations

Let's summarise the last four examples:

Linear transformations	Description	Matrix
$\mathbb{R} \to \mathbb{R}$	$f(u) = au$	$\begin{bmatrix} u \end{bmatrix} \mapsto \begin{bmatrix} a \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$
$\mathbb{R} \to \mathbb{R}^2$	$f(u) = u \begin{bmatrix} a \\ b \end{bmatrix}$	$\begin{bmatrix} u \end{bmatrix} \mapsto \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$
$\mathbb{R}^2 \to \mathbb{R}$	$f \begin{bmatrix} u \\ v \end{bmatrix} = au + bv$	$\begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$
$\mathbb{R}^2 \to \mathbb{R}^2$	$f \begin{bmatrix} u \\ v \end{bmatrix} = u \begin{bmatrix} a \\ b \end{bmatrix} + v \begin{bmatrix} c \\ d \end{bmatrix}$	$\begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

In each of the cases above, the linear transformations $\mathbb{R}^m \to \mathbb{R}^n$ above are given by $n \times m$ matrices, for $n, m \in \{1, 2\}$.

This is true in general.

Linear transformations from $\mathbb{R}^m$ to $\mathbb{R}^n$

Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a linear transformation.

Write $f \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{bmatrix}$, $f \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \end{bmatrix}$, $\ldots$, $f \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} a_{m1} \\ a_{m2} \\ \vdots \\ a_{mn} \end{bmatrix}$.

Then
$$f \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = u_1 f \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + u_2 f \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + u_m f \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$
$$= u_1 \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{bmatrix} + u_2 \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \end{bmatrix} + \cdots + u_m \begin{bmatrix} a_{m1} \\ a_{m2} \\ \vdots \\ a_{mn} \end{bmatrix}.$$
$$= \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}.$$

That is, if $x \in \mathbb{R}^m$ then $f(x) = f_A(x)$, where

$$A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

Some properties of linear transformations

Let $f : V \longrightarrow W$ be a linear transformation.

Then

- $f(0) = 0$.
- $f(-x) = -f(x)$, for all $x \in V$.

Remark: Note that the first statement says that $f(0_V) = 0_W$. That is, f has to map the zero vector of V to the zero vector of W.

Why?

- $f(0) = f(0 \cdot 0) = 0f(0) = 0$.
- $f(-x) = f(-1 \cdot x) = -f(x)$.

More properties of a linear transformation

Suppose that $f : V \longrightarrow W$ is a linear transformation and set

$$X = \{ f(x) : x \in V \} \subseteq W.$$

Claim X is a vector subspace of W.

(1) $X \neq \emptyset$ since $0 = f(0) \in X$.

(2) If $x', y' \in X$ then $x' = f(x)$ and $y' = f(y)$.
So that $x' + y' = f(x) + f(y) = f(x + y) \in X$.

(3) If $x' \in X$ and $r \in \mathbb{R}$ then $x' = f(x)$ for some $x \in V$.
So $rx' = rf(x) = f(rx) \in X$.

So X is a non-empty subset of W which is closed under vector addition and scalar multiplication.

Hence, X is a vector subspace of W.

More examples of linear transformations

$\mathbb{D}$ is the vector space of differentiable functions.

Claim : Differentiation is a linear transformation on $\mathbb{D}$.

This is the map $D : \mathbb{D} \longrightarrow \mathbb{D}$ give by $D(f) = f'$.

Check: $D(rf + sg) = \frac{d}{dx}(rf + sg) = r\frac{d}{dx}f + s\frac{d}{dx}g$ $= rf' + sg'$ That is, $D(rf + sg) = rD(f) + sD(g)$.

Similarly, integration defines a linear transformation: $I_{ab}(f) = \int_a^b f(x)\,dx$.

We have,

$$\begin{aligned} I_{ab}(rf + sg) &= \int_a^b (rf(x) + sg(x))\,dx \\ &= r\int_a^b f(x)\,dx + s\int_a^b g(x))\,dx \\ &= rI_{ab}(f) + sI_{ab}(g). \end{aligned}$$

Hence, I_{ab} is a linear transformation.