Math2061: Linear Mathematics Component

- Geoff Phillips, Carslaw 486
- Math2061 consists of two separate modules:
  Linear Mathematics and Vector Calculus.
- Math2061 web page
  www.maths.usyd.edu.au/u/UG/SS/SS2061

Lectures, tutorials, practice classes for six weeks

- Classes:

3 hours of lectures	Mon 2pm-4pm and Tues 2pm
1 Practice Session	Tues 3pm
1 Tutorial	Mon 4pm or Tues 4pm

- Consultations: by arrangement
- Assessment

65%:	Exam at end of semester 1.
10%:	Quiz (on week 3)
10%:	Quiz (on week 5)
10%:	Assignment (due week 5).
5%:	Tutorial participation.

Textbooks and references

- Textbook

  Jenny Henderson Math2061: Linear Mathematics
  Available from Kopystop
  (55 Mountain street, Broadway).

- References

  David Poole *Linear Algebra: A Modern Introduction (full version)*

  For Revision of First Year:
  David Easdown *A First Course in Linear Algebra*

  The *Little Blue Book* by Britton and Henderson is a good summary of First Year calculus and algebra.

What is *linear mathematics* about?

- Linear mathematics is one of the foundations of modern mathematics.
- It is important theoretically because so many apparently different processes in the natural world have the same *linear structure* – they are *vector spaces*.
- Many non–linear processes are often so complicated that they are modelled by linear approximations as a first step towards their understanding. Numerical solutions to many linear problems can be found quickly and accurately, using the theory from this course.
- We will study the beginnings of vector space theory and discuss some of the applications.
- Preparatory work: You should revise the following topics from Math1002 or Math1014: equations of lines and planes in space, the solution of systems of linear equations, reduced row echelon form of a matrix, elementary matrices, the calculation of eigenvectors and eigenvalues for 2×2 and 3×3 matrices.

Course objectives

- To introduce the basic concepts of vector spaces.
- To demonstrate how abstract theory can be applied to concrete problems in science and engineering.
- To develop logical thinking and ability to analyse logical arguments.

Outcomes

At the end of this unit of study you should be able to:

- solve a system of linear equations,
- apply the subspace test in several different vector spaces,
- calculate the span of a given set of vectors in various vector spaces,
- test vectors for linear independence and dependence,
- find a polynomial of minimum degree that fits a set of points exactly,

- find bases of the fundamental subspaces of a matrix,
- diagonalise an $n \times n$ matrix (for small n),
- apply diagonalisation to solve recurrence relations and systems of DEs.
- identify linear transformations between vector spaces,

Your job! For best results and maximum enjoyment in this course you should:

- make every effort to attend all the lectures, practice sessions and tutorials.
- Do as many problems as you can, and make sure you work through the solutions to the tutorial exercises each week.
- Read and understand the reference books and your own lecture notes!
- Come to the lecturers with your questions at consultation times. We are there help you.

Weekly outline of course

- Week 1: Linear systems, Gaussian elimination, vector spaces.
- Week 2: Vector spaces and subspaces, Linear combinations and linear independence, Span of a set of vectors.
- Week 3: Basis and dimension of a vector space, Linear transformations as maps between vector spaces, Column space, Null space and Rank of a matrix.
- Week 4: Geometric interpretation of matrix transformations. Eigenvalues and eigenvectors, diagonalisation of a matrix, eigenvalues of symmetric matrices.
- Week 5: Mathematical modelling and eigenvalues, Leslie population model, Linear recurrence relations.
- Week 6: Systems of linear differential equations, More on eigenvalues and eigenvectors, Review of previous examinations.

Linear equations—revision

A system of m linear equations in the unknowns $x_1, x_2, \ldots, x_n$, is a set of equations:

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n & = b_1 \\ \vdots & \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n & = b_m \end{cases}$$

This can be written as a single matrix equation:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

- Systems of linear equations can be solved using Gaussian elimination
- Gaussian elimination is the fundamental tool which underpins almost everything in this course. You should revise your notes from Math1002 or Math1014 and *make sure that you understand how it works*.

Example In the commercial preparation of nitric acid, one of the steps is the production of nitrogen dioxide:

$$NH_3 + O_2 \longrightarrow NO_2 + H_2O.$$

We want to balance this equation.

Suppose: w molecules of NH_3 yield
y molecules of NO_2
x molecules of O_2 and
z molecules of H_2O

Balancing the atoms of the various elements we get

Nitrogen	$w = y$
Hydrogen	$3w = 2z$
Oxygen	$2x = 2y + z$

This corresponds to the system of linear equations:

$$\begin{array}{rrrrcl} w & & -y & & = & 0 \\ 3w & & & -2z & = & 0 \\ & 2x & -2y & -z & = & 0 \end{array}$$

We have the system of linear equations:

$$\begin{array}{rrrrcl} w & & -y & & = & 0 \\ 3w & & & -2z & = & 0 \\ & 2x & -2y & -z & = & 0 \end{array}$$

So we have the following general solution:

$$\begin{array}{ll} w & = \frac{2}{3}t \\ x & = \frac{7}{6}t \\ y & = \frac{2}{3}t \\ z & = t \in \mathbb{R} \end{array} \qquad or \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{pmatrix} \frac{2}{3}t \\ \frac{7}{6}t \\ \frac{2}{3}t \\ t \end{pmatrix}$$

Taking $t = 6$ we get the particular solution:

$$\begin{array}{ll} w & = 4 \\ x & = 7 \\ y & = 4 \\ z & = 6 \end{array}$$

We take $z = 6$ so that the solution is *integer* valued.

Therefore, the balanced equation is:

$$4NH_3 + 7O_2 \longrightarrow 4NO_2 + 6H_2O.$$

Gaussian Elimination Example 2

Solve the equations: $\begin{cases} x & +4y & +7z & = 7 \\ 2x & +5y & +8z & = 11 \\ 3x & +6y & +9z & = 15 \end{cases}$

Again, we write down the augmented matrix and apply row operations:

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 7 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 15 \end{array}\right] \xrightarrow[R_3=R_3-3R_1]{R_2=R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 4 & 7 & 7 \\ 0 & -3 & -6 & -3 \\ 0 & -6 & -12 & -6 \end{array}\right]$$

$$\xrightarrow{R_3=R_3+2R_2} \left[\begin{array}{ccc|c} 1 & 4 & 7 & 7 \\ 0 & -3 & -6 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2=-\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 4 & 7 & 7 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the general solution is $z = t$, $y = 1 - 2z = 1 - 2t$ and $x = 7 - 4y - 7z = 3 + t$.

That is, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3+t \\ 1-2t \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}}_{\text{particular solution}} + t \underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}_{\text{general solution}}$,

where $t \in \mathbb{R}$.
How many solutions are there?

Consider the matrix equation: $Ax = b$.
How many solutions can this equation have?

From last year we know that, depending on A and b, this equation can have:

- no solutions
  E.g. $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] \Longleftrightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$
  inconsistent

- exactly one solution
  E.g. $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \end{array}\right] \Longleftrightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right] \Longleftrightarrow$
  $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

- an infinite number of solutions

  E.g. $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \iff \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1-t \\ t \end{bmatrix}$, for $t \in \mathbb{R}$.

When does a system of equations have a solution?

> **Claim 1** Suppose that $A = (a_{ij})$ is an $n \times m$ matrix. Then the matrix equation $Ax = b$ has a solution if and only if b can be written as a linear combination of the columns of A.

- Write

  $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & \cdots & a_m \end{bmatrix},$$

  where $a_i = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{bmatrix}$ is column i of A.

- We need to show that b is a solution if and only if $b = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m$, for some numbers $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$.

- First, suppose that $b = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m$.

Then, because of how matrix multiplication works, $b = \lambda_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} + \lambda_2 \begin{bmatrix} a_{21} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + \lambda_m \begin{bmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{bmatrix}$ So b is the solution corresponding to $x = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix}$

Conversely, suppose that y is a solution to $Ax = b$. Expanding the equation $b = Ay$ we discover that:

$$
\begin{aligned}
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \\
&= \begin{bmatrix} y_1 a_{11} + y_2 a_{12} + \cdots + y_m a_{1m} \\ y_1 a_{21} + y_2 a_{22} + \cdots + y_m a_{2m} \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ y_1 a_{n1} + y_2 a_{n2} + \cdots + y_m a_{nm} \end{bmatrix} \\
&= y_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} + y_2 \begin{bmatrix} a_{21} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + y_m \begin{bmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{bmatrix} \\
&= y_1 a_1 + y_2 a_2 + \cdots + y_m a_m
\end{aligned}
$$

So the solution y can be written as linear combination of the columns of A, as claimed.

We have now seen that:

- **Claim 1** Suppose that $A = (a_{ij})$ is an $n \times n$ matrix. The matrix equation $Ax = b$ has a solution if and only if b can be written as a linear combination of the columns of A.

- Going back to Example 2 we saw that the general solution of the augmented matrix equation

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 7 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 15 \end{array}\right]$$

  is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3+t \\ 1-2t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, where $t \in \mathbb{R}$.

  Reading 'backwards',

$$b = \begin{bmatrix} 7 \\ 11 \\ 15 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + 0\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$