GROUP SCHEMES AND REPRESENTATIONS

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> The very idea of scheme is of infantile simplicity—so simple, so humble, that no one before me thought of stooping so low. So childish, in short, that for years, despite all the evidence, for many of my erudite colleagues, it was really "not serious"!

Grothendieck, *Récoltes et Semailles*, quoted in [McL07]..

Introduction

The goal of this course is to gain some feeling for group schemes and their representation theory. This subject mixes group theory, algebraic geometry and representation theory. The class will not be self-contained, and I will ask you to believe some theorems and expect you to actively read other sources etc. Some background in algebraic geometry will be very helpful, but (if you are willing to do some extra reading, do exercises and take some things on faith) is not essential.[1]

Topics I hope to cover include:

(1) Basic theory of algebraic groups and group schemes.
(2) Basic structure of commutative groups schemes.
(3) Infinitesimal theory. Lie algebras. Distribution algebras.
(4) Root data, Weyl groups, affine Weyl groups.
(5) Structure of reductive algebraic groups.
(6) Basic representation theory of reductive algebraic groups.
(7) Steinberg theorems.

Below I have included a bibliography of the sources I will be referring to during this course. The most important references are the books of Springer, Waterhouse, Bourbaki and Jantzen (this is roughly the order in which these references will be relevant).

Questions / office hours

Please do *not* simply knock unannounced on my door! My e-mail address and office number is below. I am happy to answer questions about this course via e-mail, and to make appointments to discuss in detail.

[1] For a very clear discussion of what modern algebraic geometry "is" see the introduction to [GD71].

Student talks

Usually this course will run from 3pm - 5pm on Thursdays. However I would like to have some lectures by interested parties on subjects on the fringe (e.g. historical motivation). I imagine on such days I will finish a little earlier (say 4:45pm), we will have a short break, and then interested parties will reconvene for an hour or so. Here are some interesting topics (in roughly increasing order of necessary mathematical background) but you are welcome to suggest your own:

(1) Discuss in detail the finite-dimensional representation theory of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.
(2) Discuss some examples of Lie groups (e.g. S^1, $SO(3)$, $\dots$) and why they are important.
(3) Discuss in detail the famous[2] letter of Galois (ability to read French required).
(4) Discuss Fourier series and the Fourier transform from the point of view of the representation theory of the Lie groups S^1 and $\mathbb{R}$.
(5) Discuss finite groups of Lie type. What is their "rough" structure? Why are they often (almost) simple groups?
(6) Discuss Riemann's classic work on the foundations of geometry [Rie13].
(7) Discuss the notion of a smooth (or "simple") point on an algebraic variety, basically following the introduction to Zariski's classic [Zar47].
(8) Give a historical introduction to the development of Lie groups [Haw00]. What was a Lie group for Lie? for Killing? Why was Lie so interested?

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[2] André Weil remarked that this was one of the most important letters in the history of humanity. He was clearly biased, but there is little doubt that the letter has some merit!