Very useful exercise:

Let V be a $\mathbb{G}_m$-module.

(a) The comodule structure is given by a map

$$\begin{aligned} V \xrightarrow{c} V \otimes k[\mathbb{G}_m] &= V \otimes k[X, X^{-1}] \\ &= \dots \oplus (V \otimes kX^{-1}) \oplus (V \otimes k) \oplus (V \otimes kX) \oplus \dots \\ &= \dots \oplus VX^{-1} \oplus V \oplus VX \oplus \dots \end{aligned}$$

hence is given by maps $\phi_i : V \longrightarrow V \cancel{\otimes kX^i}$ for all $i \in \mathbb{Z}$.

s.t. $c(v) = \sum_{i \in \mathbb{Z}} \phi_i(v) \otimes X^i$ for all v.

(b) By considering the diagram:

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V \otimes k[\mathbb{G}_m] \arrow[d] \\ V \otimes k[\mathbb{G}_m] \arrow[r] & V \otimes k[\mathbb{G}_m] \otimes k[\mathbb{G}_m] \end{tikzcd}$$

Show that $\phi_i^2 = \phi_i$ and that $\phi_i \phi_j = 0$ if $i \neq j$.

(c) Deduce that $id_V = \sum_{i \in \mathbb{Z}} \phi_i$ and that

$$V = \bigoplus_{i \in \mathbb{Z}} \phi_i(V) .$$

(d) Hence show that a $\mathbb{G}_m$-module is the same thing as a $\mathbb{Z}$-grading on V.