~~Schemes~~

Group Schemes and Representations:

Structure of the course. Office hours etc. Please ask questions. Student talks.

I will omit many proofs but try to give lots of background.

Please send me an e-mail.

Historical Background for this course:

Evariste Galois (1832): importance of the group concept.

$$\begin{array}{c} E \\ | \\ F \end{array} \quad \begin{array}{l}\text{finite} \\ \text{Galois} \\ \text{extension}\end{array} \qquad \text{intermediate fields} \overset{\sim}{\longleftrightarrow} \text{subgroups of } \overset{\Gamma}{\overset{\shortparallel}{\mathrm{Gal}(E/F)}}$$

moreover structure of Γ determines structure of extension Γ.

Galois also introduced the notion of a simple group (i.e. $N \triangleleft \Gamma$ normal $\Rightarrow N = \Gamma$ or $\{1\}$),

and showed that A_n $(n \geq 5)$ / $PSL_2(\mathbb{F}_p)$ is simple, if $p \neq 2, 3$.

Camille Jordan (1870): Studied the classical groups over finite and other fields. Shows that $PSL_n(\mathbb{F}_q)$ is simple.

Felix Klein (1872): erlangen program: "geometry is its group of symmetries".

E.g.

$$\begin{array}{ccl} PGL_3(\mathbb{R}) & \mathbb{RP}^2 & \text{projective geometry (incidence, crossratio)} \\ \cup & \cup & \\ \mathrm{Aff}(\mathbb{R}^2) & \mathbb{R}^2 & \text{affine geometry (parallelism, incidence)} \\ \cup & \cup & \\ O(\mathbb{R}^2) & \mathbb{R}^2 \text{ with origin} & \text{Euclidean geometry (angles, distance)} \end{array}$$

Möbius transformations

$$\begin{array}{ccc} SL_2(\mathbb{C}) & \circlearrowright & \mathbb{P}^1\mathbb{C} \\ \cup & & \cup \\ SL_2(\mathbb{R}) & \circlearrowright & \mathbb{H} \end{array}$$

(hyperbolic geometry)

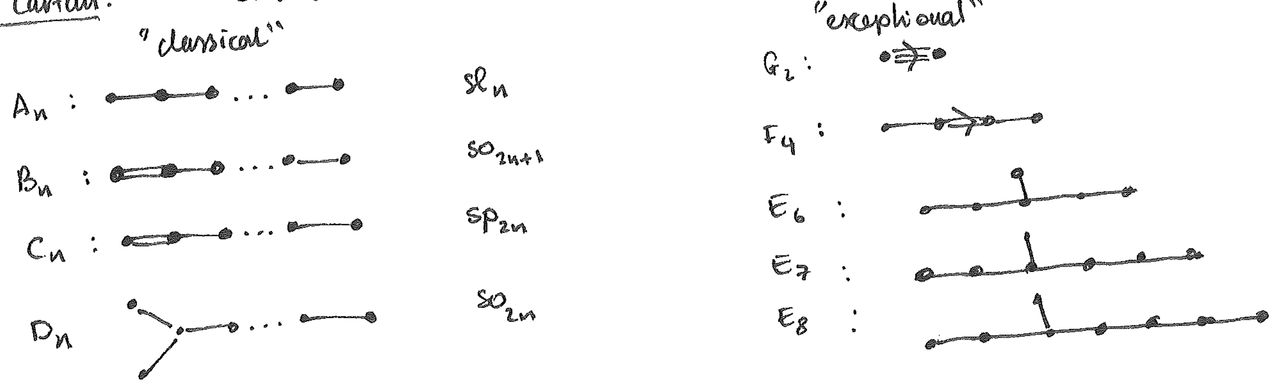
Sophus Lie: (1873-74) General study of continuous ~~sy~~ groups of symmetry.
Eg: rigid motions of 3 space.

Idée fixe: wanted a "Galois theory of differential equations".

Lie algebra: "one can write any continuous transformation as a infinite combination of infinitesimal transformations".

(1880) (1894)

Killing-Cartan: Classification of the ~~[illegible]~~ complex semi-simple Lie algebras.



(This will show up later in the course).

(Theory much more algebraic than Lie.)

Chevalley (1955): Any compact Lie group (like SO_3, Sp_4) can be "defined over $\mathbb{Z}$".

One obtains in this way a uniform approach to the finite groups of Lie type.

1960 — ? : Classification of finite simple groups: except for finitely many (~26) exceptions, all finite simple groups are either alternating groups or finite groups of Lie type.

Borel (1954): Groupes linéaires algébriques: first definitive treatment of algebraic groups.

Finally, in number theory "Lie groups over p-adic fields" (e.g. $SL_n(\mathbb{Q}_p)$) are enormously important. Here the algebraic theory is essential.

Fundamental examples: "Groups are too complicated." need algebraic theory.

Throughout k is a field. Often we will assume k algebraically closed.

How can we produce a group out of k?

Additive group: set k equipped with addition: $(x,y) \mapsto x+y$. $\mathbb{G}_a$

additive inverse: $x \mapsto -x$

Multiplicative group: set $k^\times$ equipped with multiplication: $(x,y) \mapsto xy$.

and multiplicative inverse: $x \mapsto x^{-1}$. $\mathbb{G}_m$

(Q: why is $x \mapsto x^{-1}$ considered algebraic on $k^\times$?)

n^{th} roots of unity: ~~Elements of order n~~ subgroup of $k^\times$ defined by the equation $x^n = 1$.

μ_n

Rmk: If $k = \mathbb{R}$ or $\mathbb{C}$ we have $\exp(x) = \sum \frac{x^n}{n!}$ which satisfies $\exp(x+y) = \exp(x)\exp(y)$ and hence defines a homomorphism $k \to k^\times$.

This will be forbidden in the theory, as exp is not algebraic.

(makes no sense for an arbitrary field).

One of the first theorems in the subject will be that $\mathbb{G}_m$ and $\mathbb{G}_a$ "don't mix"!

SL_2: Consider: $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; ad-bc = 1, \; a,b,c,d \in k \right\}$

with multiplication: $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) \mapsto \begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{pmatrix}$

and inverse $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

GL_2: The same as $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; ad-bc \neq 0 \right\}$ with mult. as above

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

(Q: why considered algebraic?)

The circle: (aka SO_2.) Show that $\{(x,y) \mid x^2+y^2=1\}$ $\in k \times k$

(Exercise)

can be made into an algebraic group

with inverse operation $(x,y) \mapsto (x,-y)$.

Gl_n, Sl_n:

Rmk: ① $\mathbb{G}_a, \mathbb{G}_m, \mu_n$ look trivial. Don't be deceived. They are not!

② It is a good idea to go Gl_1, Gl_2, Gl_n, not Gl_1, Gl_n!

Very important point: k doesn't really matter above.

We can view the above examples as universal recipes for producing groups:

input:		output:
field k (or even ~~k-algebra~~ ring R)	$\rightsquigarrow$	a group $G(R)$

Moreover, whenever we have a map of rings $R \to R'$ w/ 1,

get a morphism $G(R) \to G(R')$.

In other words, the above examples provide functors

$$G: \text{category of rings} \longrightarrow \text{category of groups}.$$

It is very profitable to think about an algebraic group as a whole family of groups, rather than as one group.

Affine alg. geometry: study closed subsets of k^I (I some index set) given by polynomial equations.

I.e. we fix polynomials $\{f_k\} \subset k[x_i]_{i\in X}$ and consider

$$V(\{f_i\}) = \{x \in k^I \mid f_i(x) = 0 \ \forall k\} \subset \mathbb{A}^I = k^I$$

(affine space)

It is a general principle in mathematics that ~~we~~ we shouldn't study "embedded objects".

Eg: ① every finite group is a subgroup of some symmetric group, but it is better to study "finite groups" rather than "subgroups of S_n".

⊛

②

~~Usually we write $\mathbb{A}^I$~~ If I is finite, $|I| = n$ we write $\mathbb{A}^n = k^n$ ("affine space")

Example: Consider the algebraic sets k and $P := \{(x,y) \mid y = x^2\} \subset \mathbb{A}^2$

These are "algebraically isomorphic" via $x \mapsto (x, x^2)$. Hence should be "the same".

Example: What are the algebraic subsets of k?

Example: Is $k^\times$ an algebraic subset? More generally, if $D(f)$ is an algebraic subset.

Fundamental idea: Instead of studying $X \subset \mathbb{A}^n$ (an algebraic set) we study

$$k[X] = \text{"algebraic functions on } X\text{"} := k[x_i] / (\text{functions vanishing on } X).$$

~~Rmk: ① Given $k[X]$ and the images of x_i in $k[X]$ we can recover.~~

Example: $k[\mathbb{A}^1] = k[x] \xrightarrow{\sim} k[P] = k[x,y]/(y - x^2)$ via $x \mapsto x$.

(So in this case isomorphisms of ~~rings~~ k-algebras captures the idea that we want)

Hilbert Nullstellensatz: (not so easy theorem)

If $k = \bar{k}$ (i.e. k algebraically closed) we have a bijection

$$\begin{array}{ccccc} X & \longleftrightarrow & I(X) & \longleftrightarrow & k[x]/(I) \\ \cap & & \cap & & \\ \left\{ \begin{array}{c} \text{algebraic subsets} \\ \text{of } k^n \end{array} \right\} & & \left\{ \begin{array}{c} \text{radical ideals} \\ I \subset k[x_1,\dots,x_n] \end{array} \right\} & & \left\{ \begin{array}{c} \text{reduced } k\text{-algebras} \\ B \text{ equipped w/ a} \\ \text{surjection} \\ k[x_1,\dots,x_n] \twoheadrightarrow B \end{array} \right\} \Big/ \cong \\ & & \ni & & \\ V(I) & \longleftrightarrow & I & & \end{array}$$

$\left(\text{if } f^n \in I \Rightarrow f \in I \right)$

Rmk: Forgetting the surjection $k[x_1,\dots,x_n] \twoheadrightarrow A$ is the same as forgetting the embedding $X \subset \mathbb{A}^n$.

Rmk: There is now a menagerie of adjectives which one can get used to.

Exercise: Make a list of algebraic and geometric properties and try to connect the two sides. This isn't always easy:

geometric property	algebraic property
connected	integral domain
irreducible	normal
smooth, singular pt.	field of fractions
of dimension 7	PID
open subset	localisation
	finitely generated

In fact we have an equivalence of categories:

$$\left\{ \begin{array}{c} \text{alg. varieties} \\ \text{w/ alg. morphisms} \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{commutative reduced} \\ k\text{-algebras of finite type} \end{array} \right\}^{op}.$$

Affine schemes: fundamental idea of Grothendieck.

Any commutative ring can be regarded as the functions on a "space", and this is the correct (local) setting for algebraic geometry.

In this course we will work almost exclusively with k-algebras A, but the theory is more general.

$$A \rightsquigarrow \operatorname{Spec} A = \{ \mathfrak{p} \subset A \mid \text{prime ideals} \}.$$

$\operatorname{Spec} A$ is equipped with a topology for which the basic open sets are
$$D(f) = \{ \mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p} \},$$

and a sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$ s.t.

- $\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A[\tfrac{1}{f}]$.
- $\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A = k[\operatorname{Spec} A]$
- $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} = A_{\mathfrak{p}}$ (local ring at $\mathfrak{p}$).

We have an equivalence of categories:

GA

$$\begin{array}{ccc}
\{\text{affine schemes over } k\} & \xrightarrow{\sim} & \left(\begin{array}{c}\text{commutative}\\ k \text{ algebras}\end{array}\right)^{op} \\
\cup & & \cup \\
\text{affine varieties} & \xrightarrow{\sim} & \left(\begin{array}{c}\text{reduced}\\ \text{commutative}\\ k\text{-algebras}\\ \text{of finite type}\end{array}\right)
\end{array}$$

Exercise Keep adding to your list, now in the world of schemes!

The category of affine schemes has ~~these~~ products:

Given affine schemes X, Y there exists an affine scheme $X \times Y$ and a natural ~~sets~~ iso:

$$\mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y) \xrightarrow{\sim} \mathrm{Hom}(Z, X \times Y)$$

Moreover $X \times Y$ is unique up to unique isomorphism.

We can use the equivalence G⟺A to turn this around:

$A = k[X]$, $B = k[Y]$.

$$\mathrm{Hom}(\cancel{k[X]}\, A, C) \times \mathrm{Hom}(B, C) \xrightarrow{\sim} \mathrm{Hom}(\, ?\, , C).$$

Such an object is given by the tensor product $A \otimes_k B$ of commutative ~~k~~ rings.

Exercises: (a) Learn what a tensor product is (if you don't already know)

(b) Show that $k[x] \otimes k[y] \cong k[x, y]$.

(c) Compute $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$.

(d) Suppose k is a field of char. p. Let $k(x) \subset k(x^p)$. Compute $k(x^p) \otimes_{k(x^p)} k(x)$.