Lightning overview of structure theory of algebraic groups

For this section only we allow <u>non-affine</u> algebraic groups.

I.e. G is a group object in the category of varieties $/k$.
"
(reduced, separated scheme of finite type).

We also assume $k = \bar{k}$.

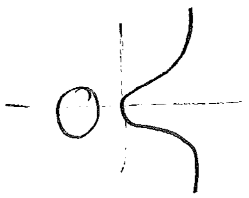
<u>Easy theorem:</u> There exists a unique connected normal subgroup $G^\circ \subset G$ s.t. G/G° is an étale group scheme (i.e. here $\cong \Gamma$ for some finite group Γ).

$$G^\circ \hookrightarrow G \twoheadrightarrow G/G^\circ .$$

Hence we can assume G is <u>connected</u>.

<u>Def:</u> An <u>abelian variety</u> is a connected proper algebraic group.

<u>Eg:</u> $\{(x:y:z) \in \mathbb{P}^2 \mid y^2 = x(x-z)(x+z)\}$ is an abelian variety "elliptic curve".



$\subset \mathbb{P}^2$

<u>Thm:</u> ① Any abelian variety is projective.

② The group law on any abelian variety is abelian.

<u>Rmk:</u> Abelian varieties are amongst the most important objects in arithmetic geometry, number theory and complex analysis.

<u>Theorem (Chevalley)</u> Any connected algebraic group G admits a unique normal subgroup H s.t. H is affine and G/H is projective. (an abelian variety)

$$H \hookrightarrow G \twoheadrightarrow G/H.$$

<u>Verbal:</u> This splits the study of algebraic groups into $\nearrow$ affine; $\searrow$ abelian varieties.

From now on we may assume that G is <u>affine</u>.

For any algebraic group, G, $[G,G] \subset G$ is a closed normal subgroup.

<u>Def:</u> An algebraic group is <u>solvable</u> if its derived series, defined by $G^{(0)} := G$, $G^{(i)} := [G^{(i-1)}, G^{(i-1)}]$, terminates in the trivial group (i.e. $G^{(m)} = \{1\}$ for some m).

<u>Theorem of Lie-Kolchin:</u> If G is solvable then G is isomorphic to a closed subgroup of T_n (the upper triangular matrices inside GL_n) for some n.

<u>Def:</u> An algebraic group is <u>unipotent</u> if it is a successive extension of products of $\mathbb{G}_a$'s.

<u>Theorem:</u> G is unipotent $\iff$ G is a closed subgroup of $U_n :=$ strictly upper triangular matrices in GL_n for some n.

Rmk: Solvable and unipotent groups "gibt's wie Sand am ~~Strand~~ Meer."

Somewhat analagous to classifying p-groups.

Theorem: If G is connected, affine then there exist unique closed normal maximal subgroups RG "radical" solvable
$R_u G$ "unipotent radical"

and exact sequences

$$RG \longrightarrow G \longrightarrow G/RG \quad \text{s.t.} \quad R/RG \text{ is } \underline{\text{semi-simple}},$$

$$R_u G \longrightarrow G \longrightarrow R/R_u G \quad R/R_u G \text{ is } \underline{\text{reductive}}.$$

G is semi-simple (reductive) if $RG = \{1\}$ (resp. $R_u G = \{1\}$).

It turns out that semi-simple and reductive groups admit a classification via discrete data "root data", this is what we want to explain now.

Thm: If char $k = 0$ and G/k is reductive then Rep G is semi-simple.

(I think this is where the name comes from).

Thm: Assume that G admits a faithful irreducible representation, then G is reductive.

Ex: ~~Sl_2, Sp_{2n} are~~ Gl_{2n}, Sl_n, Sp_{2n}, SO_{2n} are reductive.

Some examples of reductive groups:

① Gl_n, Sl_n (natural representation is faithful, irreducible) ~~and~~

let $J_n = \underbrace{\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}}_{n} \Big\} n$ anti-diagonal matrix.

② Define $O_n = \{ X \in Gl_n \mid X^t J X = J \}$. "orthogonal group"

$SO_n = \{ X \in Gl_n \mid X^t J X = J, \ \det X = 1 \}$. "special orthogonal group"

③ ~~to~~ let $\tilde{J} = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$.

$$Sp_{2n} = \{ X \in Gl_{2n} \mid X^t \tilde{J} X = \tilde{J} \}.$$

<u>Exercise:</u> Show that we can take any non-singular matrix above in place of J. The above choice makes the calculation of the Lie algebra particularly pleasant, as we will see later.

Root Systems + Root Data

Let V be a f.d. real vector space. A reflexion is an endomorphism s such that $s-1$ is of rank 1, and $s^2 = id$. (i.e. eigenvalues are $(-1, 1, 1, \ldots)$).

Any reflexion is of the form $\underset{s_{\alpha,\alpha^\vee}:}{} \ v \mapsto v - \langle \alpha^\vee, v\rangle \alpha$ for $\alpha \in V$, $\alpha^\vee \in V^*$, s.t. $\langle \alpha^\vee, \alpha\rangle = 2$, unique up to $\alpha \mapsto \lambda\alpha$, $\alpha^\vee \mapsto \lambda^{-1}\alpha^\vee$.

Lovely Lemma: Suppose $R \subset V$ is finite, $\langle R\rangle = V$. For $\alpha \in R$ there is at most one reflection s s.t. $s(R) = R$ and $s(\alpha) = -\alpha$.

Proof: Let s, s' be two such reflections. Then $t = ss'$ satisfies $t(\alpha) = s(s'(\alpha)) = \alpha$, $t(R) = R$. Hence t is of finite order. Now s, s' act as the identity on $V/\langle\alpha\rangle$ and hence so does t. Hence there exists $\gamma \in V^*$ s.t. $t(x) = x + \gamma(x)\alpha$, for all $x \in V$. Now

$t^n(x) = x + n\gamma(x)\alpha$, hence $\gamma = 0$ and $t = id$. □

Def: A root system is a subset $R \subset V$ s.t.

① R is finite, $0 \notin R$, and $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$ for all $\alpha \in R$.

② for any α, $\exists \alpha^\vee \in V^*$ s.t. $s_{\alpha,\alpha^\vee}(R) \subset R$. ($\alpha^\vee$ is unique after Lemma).

③ $\alpha^\vee(R) \subset \mathbb{Z}$ ("integrality")

Given root systems $R_1 \subset V_1$, $R_2 \subset V_2$ their direct sum is $R_1 \cup R_2 \subset V_1 \oplus V_2$.

A root system is irreducible if it is not isomorphic to a non-trivial direct sum. Alternatively if $\nexists!$ $V = V_1 \oplus V_2$ with $R = (V_1 \cap R) \cup (V_2 \cap R)$.

The rank of a root system (V,R) is $\dim V$.

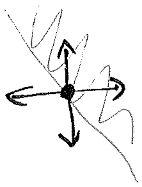
Examples:

(a) Rank 1: There exists a unique root system: $R=\{\pm\alpha\}\subset V=\mathbb{R}$, $\alpha\neq 0$.

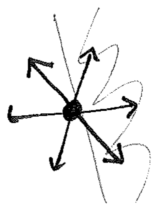
"A_1".

(b) Rank 2: Here are four examples:

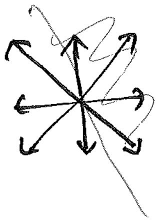
$A_1\times A_1$



A_2



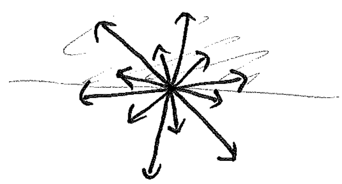
B_2



$\{e_1,e_2\}\cup\{\pm e_1\pm e_2\}$.

$\{\pm e_i \pm e_j \mid i\neq j\}\subset \mathbb{R}e_1\oplus\mathbb{R}e_2\oplus\mathbb{R}e_3/(e_1+e_2+e_3=0)$.

G_2



Exercise: These are all.

A system of positive roots is a subset $R^+\subset R$ s.t.

① $R=R^+\cup\{-R^+\}$ ② if $\alpha,\beta\in R^+$ then either $\alpha+\beta\in R^+$ or $\alpha+\beta\notin R$.

Proposition: Given any functional $\gamma: V\to\mathbb{R}$ s.t. $\gamma(R)\not\ni 0$,
$R^+=\{\alpha\in R \mid \gamma(\alpha)>0\}$ is a system of positive roots.

Proposition: For any system R^+ of positive roots, consider
$$\Sigma=\{\alpha\in R^+ \mid \alpha\neq\alpha_1+\alpha_2,\ \alpha_1,\alpha_2\in R^+\}.$$
Then Σ is a basis of V and $R^+=\mathbb{Z}_{\geq 0}\Sigma\cap R$.

Σ is called a basis of simple roots.

Consider $R \subset V$ a root system.

Let R^+ denote a system of positive roots and Σ the corresponding basis of simple roots.

We associate a <u>Dynkin diagram</u> to Σ as follows:

- vertices are the elements of Σ.
- edges are of 3-types:

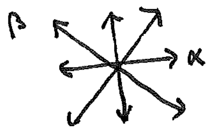
consider $\alpha, \beta \in \Sigma$, $V_{\alpha,\beta} := \langle \alpha, \beta \rangle$, $R_{\alpha,\beta} := R \cap V_{\alpha,\beta}$

$(R_{\alpha,\beta} \subset V_{\alpha,\beta})$ is a rank 2 root system.

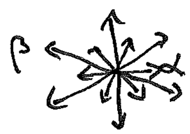
Four cases:

$A_1 \times A_1$: no edge.

A_2: 

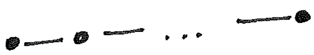
B_2: 

: 

G_2: 

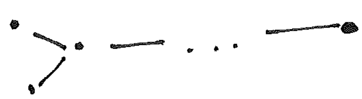
<u>Thm</u>: The Dynkin diagram does not depend on choice of $R \subset V$ and determines the isomorphism class of R.

The Dynkin diagrams of the irreducible root systems are as follows:

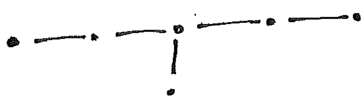
A_n: 

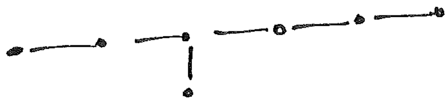
B_n: 

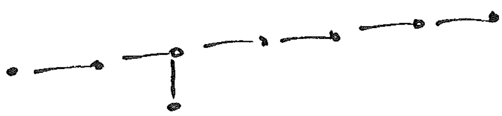
C_n: 

D_n: 

F_4: 

E_6: 

E_7: 

E_8: 

G_2: 

A <u>root datum</u> is a quadruple $(\mathcal{X}, R, \mathcal{X}^\vee, R^\vee)$ where

① $\mathcal{X}, \mathcal{X}^\vee$ are free finite rank $\mathbb{Z}$-modules eqipped with a perfect pairing
$$\langle -,- \rangle : \mathcal{X} \times \mathcal{X}^\vee \to \mathbb{Z}.$$

② $R \subset \mathcal{X}$, $R^\vee \subset \mathcal{X}^\vee$ are finite sets equipped with a bijection $R \ni \alpha \mapsto \alpha^\vee \in R^\vee$;

satisfying

ⓐ $\langle \alpha^\vee, \alpha \rangle = 2$ and $\mathbb{Z}\alpha \cap R = \{\pm\alpha\}$.

ⓑ $\forall \alpha \in R$ the maps $s_\alpha: \lambda \mapsto \lambda - \langle \alpha^\vee, \lambda \rangle \alpha$, (resp. $s_\alpha^\vee: \lambda^\vee \mapsto \lambda^\vee - \langle \lambda, \alpha \rangle \alpha^\vee$

preserve R (resp. $R^\vee$).)

The rank of a root datum is $\mathrm{rk}\, \mathcal{X} = \mathrm{rk}\, \mathcal{X}^\vee$.

<u>Example:</u> There are 3 root datum of rk 1:

ⓐ $\mathcal{X} = \mathcal{X}^\vee = \mathbb{Z}$, $R = R^\vee = \emptyset$. } not semi-simple.

ⓑ " $R = \{\pm 2\}$, $R^\vee = \{\pm 1\}$.

ⓒ " $R = \{\pm 1\}$, $R^\vee = \{\pm 2\}$.

$\Big\}$ semi-simple

Given a root datum we can consider
$$R \subset \langle R \rangle \otimes_{\mathbb{Z}} \mathbb{R}.$$
This is a root system.

If $\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{R} = \langle R \rangle_{\mathbb{R}}$ we say our root system is semi-simple.

Assume we are in the <u>semi-simple</u> case, that is $\langle R\rangle_{\mathbb{R}} = V := \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{R}$.

Proposition: Fix a root system (V, R)
We have a bijection:

$$\left\{ \begin{array}{l} \text{root data with} \\ \text{root system } (V,R) \end{array} \right\} \Big/ \cong \quad \longrightarrow \quad \text{subgroups } P/Q \quad \text{where}$$

$P = \mathbb{Z}R$ "root lattice"

$Q = \{ \lambda \in V^{*} \mid \langle \alpha^\vee, \lambda \rangle \in \mathbb{Z} \}$ "weight lattice".

Examples: ① If (V, R) is of type A_1, i.e. $V = \mathbb{R}$, $R = \{\pm\alpha\}$

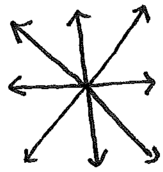
then $\alpha^\vee \in V^*$ is given by $\langle \alpha^\vee, \alpha \rangle = 2$. Hence

$P = \mathbb{Z}\alpha$. $\quad Q = \frac{1}{2}\mathbb{Z}\alpha$. $\quad Q/P = \mathbb{Z}/2\mathbb{Z}$.

Two root data have: $\mathcal{X} = \mathbb{Z}\alpha$, $\mathcal{X}^\vee = \frac{1}{2}\mathbb{Z}\alpha^\vee$ "PGL_2".

$\mathcal{X} = \frac{1}{2}\mathbb{Z}\alpha$, $\mathcal{X}^\vee = \mathbb{Z}\alpha^\vee$ "SL_2".

② Suppose (V, R) is of type B_2, i.e. $V = \{\pm e_i \mid i = 1,2\} \cup \{\pm e_1 \pm e_2\}$.



Then $R^\vee = \{\pm 2e_1, \pm 2e_2, \pm e_1 \pm e_2\} \subset V^*$. Hence

$P = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$

$Q = \langle e_1, \frac{1}{2}(e_1 + e_2) \rangle$. Hence P/Q $\cancel{Q/P} = \mathbb{Z}/2\mathbb{Z}$.

$A_2 \quad P/Q = \mathbb{Z}/3$.

③ <u>Exercise:</u> If V is of type G_2 then $P/Q = \{1\}$.