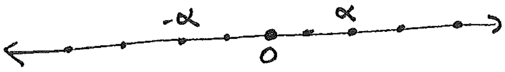
Last week: $(\mathcal{X}, R, \mathcal{X}^\vee, R^\vee)$ root datum.

$\langle -,- \rangle : \mathcal{X}^\vee \times \mathcal{X} \to \mathbb{Z}$ perfect pairing, $R \subset \mathcal{X}$ finite "roots", ...

$$s_\alpha(\lambda) := \lambda - \langle \alpha^\vee, \lambda \rangle \alpha : \mathcal{X} \to \mathcal{X} \quad \text{s.t.} \quad s_\alpha(R) \subset R \text{ etc.}$$

<u>Basic examples:</u> ① $\mathcal{X} = \mathcal{X}^\vee = \mathbb{Z}$. $R = \underbrace{\{\pm 2\}}_{=\{\pm\alpha\}}$. $R^\vee = \{\pm 1\}$.



② $\mathcal{X} = \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i$, $\mathcal{X}^\vee = \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i^*$, $R = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$, $R^\vee = \{\varepsilon_i^* - \varepsilon_j^* \mid i \neq j\}$.

<u>Important remark:</u> The definition of root datum is symmetric in $R \subset \mathcal{X}$ and $R^\vee \subset \mathcal{X}^\vee$! Swapping is at the heart of Langlands duality.

<u>Today:</u> how to go from a reductive group to its root datum.

<u>First:</u> <u>diagonalisable groups.</u>

Recall $\mathbb{G}_m$: $k[\mathbb{G}_m] = k[X^{\pm 1}]$, $\Delta(X) = X \otimes X$.

For any G, $a \in k[G]$ is <u>group-like</u> if $\Delta(a) = a \otimes a$,

$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} \Rightarrow$
$\Downarrow$
$\varepsilon(a) = 1$

<u>Observe:</u> $\mathrm{mult}\,(\mathrm{id} \otimes S)(\Delta) = \varepsilon$, hence $S(a) = a^{-1}$, for a group-like elt.

Also: $\Delta(ab) = \Delta(a)\Delta(b)$, hence group like elts form a group.

Hence: $\mathrm{Hom}(G, \mathbb{G}_m) = \mathrm{Hom}_{\mathrm{Hopf}}(k[\mathbb{G}_m], k[G]) = \{\text{group like elts}\} \subset k[G]$.
$\quad \mathcal{X}(G) = \mathrm{Hom}(G, \mathbb{G}_m)$

<u>Exercise:</u> The set of group like elts is linearly independent.

A group scheme G/k is <u>diagonalisable</u> if $k[G]$ is spanned by group like elements, $\mathfrak{X}(G)$.

E.g. $k(\mathbb{G}_m) = \bigoplus_{n\in\mathbb{Z}} kX^n$ and $\Delta(X^n) = X^n \otimes X^n$. Hence $\mathbb{G}_m$ is diagonalisable.

Thm: (not difficult)

- If G is <u>diagonalisable</u> then $k[G] = k[\mathfrak{X}(G)]$.
- $\operatorname{Rep} G \xrightarrow{\sim} \mathfrak{X}(G)$-graded vector spaces.

  In particular, any Rep G is semi-simple.

- If G is a finite type, then $G \hookrightarrow \mathrm{Diag} \subset \mathrm{Gl}_n$ for some n. (hence "diagonalisable").
- G is diagonalisable $\iff$ Rep G is semi-simple and all simples are of dimension 1.
- $G \mapsto \mathfrak{X}(G)$ provides an anti-equivalence:

$$\begin{array}{ccc} \text{diag. groups} & \longleftrightarrow & \text{abelian groups} \\ \cup & & \cup \\ \text{finite type} & \longleftrightarrow & \text{fin.g.} \\ \cup & & \cup \\ \text{connected} & \longleftrightarrow & \text{free} \end{array}$$

Now we assume G is a reductive algebraic group $/k = \bar{k}$.

Def: • A <u>Borel subgroup</u> is a maximal, closed, solvable subgroup of G.

- A torus is a closed connected diagonalisable subgroup ($\cong \mathbb{G}_m^n$).
- A <u>maximal</u> torus is a torus not properly contained in any other torus.

<u>Exercise:</u> ① $Z_{\mathrm{Gl}_n}(\mathrm{diag}) = \mathrm{diag}$, hence $\mathrm{diag} \subset \mathrm{Gl}_n$ is a maximal torus.

② A Borel subgroup in Gl_n is $\begin{pmatrix} \ddots & * \\ & \ddots \end{pmatrix}$.

Rmk: (should come after theorem).

If $\mathcal{X}(G)$ is f.g. free then $\mathcal{X}(G) \cong \mathbb{Z}^n$ for some n,

hence $k[G] = k[\mathcal{X}(G)] = k[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$

and so $G \cong \mathbb{G}_m^n$.

In this case $\mathrm{Hom}(\mathbb{G}_m, G) = \mathcal{X}^\vee(G) \cong \mathbb{Z}^n$,

and $\mathcal{X}^\vee(G) \times \mathcal{X}(G) \to \underset{\mathcal{X}(\mathbb{G}_m)}{\overset{\shortparallel}{\mathbb{Z}}}$ is a perfect pairing.

Thm: Any two maximal tori / Borel subgroups are conjugate in G

Now fix (G,T) (reductive group, maximal torus).

G acts by conjugation on G, hence on Lie G "adjoint rep".

preserving 1

Set $\mathcal{X} := \mathcal{X}(T)$, $\mathcal{X}^\vee := \mathcal{X}^\vee(T)$, perfect pairing $\mathcal{X}^\vee \times \mathcal{X} \to \mathbb{Z}$.

How to get R?

Thm: As T-modules we have

α wt space for T-action.

$$\mathrm{Lie}\, G = \mathrm{Lie}\, T \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

for a subset $R \subset \mathcal{X}(T)$ s.t. $|R| < \infty$, $R \cap \{0\} = \emptyset$, $R = -R$.

Moreover, $\dim \mathfrak{g}_\alpha = 1$.

Do example of GL_n simultaneously on the board.

How to get $R^\vee \subset \mathcal{X}^\vee$?

$\boxed{\text{Slogan: Any reductive group is "built out of } SL_2\text{'s", root data tells us how to assemble them.}}$

For any $\alpha \in R$ there exists $\varphi_\alpha : \mathbb{G}_a \longrightarrow G$ ("one parameter subgroup")

s.t. $d\varphi_\alpha : \operatorname{Lie}\mathbb{G}_a = k \longrightarrow \mathfrak{g}_\alpha$ is an isomorphism.

We can extend φ to a map $\phi_\alpha : SL_2 \longrightarrow G$ s.t. $\varphi = \phi\begin{pmatrix}1 & \lambda \\ 0 & 1\end{pmatrix}$.

ϕ $\qquad$ $\phi_\alpha\begin{pmatrix}\lambda & 0 \\ 0 & \lambda^{-1}\end{pmatrix} \subset T$.

Composing $\lambda \mapsto \begin{pmatrix}\lambda & 0 \\ 0 & \lambda^{-1}\end{pmatrix} \longrightarrow G$ gives $\alpha^\vee : \mathbb{G}_m \longrightarrow T$.

<u>Example:</u> (1) $G = SL_2$, $T = \begin{pmatrix}\lambda & 0 \\ 0 & \lambda^{-1}\end{pmatrix}$. $\mathcal{X}(T) = \mathbb{Z}\varepsilon$

where $\varepsilon\begin{pmatrix}\lambda & 0 \\ 0 & \lambda^{-1}\end{pmatrix} = \lambda$.

$T \circlearrowright \mathfrak{sl}_2$ via $\begin{pmatrix}\lambda & 0 \\ 0 & \lambda^{-1}\end{pmatrix}\begin{pmatrix}a & b \\ c & -a\end{pmatrix}\begin{pmatrix}\lambda & 0 \\ 0 & \lambda^{-1}\end{pmatrix} = \begin{pmatrix}a & \lambda^2 b \\ \lambda^{-2} c & -a\end{pmatrix}$.

Hence $\alpha = \{\pm 2\varepsilon\}$. If $\alpha = 2\varepsilon$ then $\phi_\alpha : SL_2 \longrightarrow SL_2$

can be taken to be the identity. Gives $\alpha^\vee = \varepsilon^* \in \mathcal{X}^\vee(T)$.

(2)

<u>Theorem:</u> (Isomorphism and existence). The above construction gives a bijection.

$$\left\{\begin{array}{l}\text{Reductive groups}\\ \text{up to isomorphism}\end{array}\right\} \xleftrightarrow{\sim} \text{Root Data}/\cong .$$

Choices of positive roots $\longleftrightarrow$ Borel subgroups ~~T~~ B, containing T.

<u>Rmk:</u> Classification is independent of (algebraically closed) base field k.

Example: $G = SL_n$. (PERHAPS BETTER TO DO GL_n).

$$\mathrm{Lie}\, G = \{ (id + \varepsilon X) \mid \det(id + \varepsilon X) = 1 \}$$
$$= \{ X \mid \mathrm{Tr}\, X = 0 \}.$$

$T = \mathrm{diag} \cap SL_n$ is a maximal torus. (Exercise.)

$\mathrm{Lie}\, T =$ diagonal matrices. Note that:

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{array}{c} \\ i \to \end{array}\!\!\begin{array}{c} j \\ \downarrow \\ \begin{pmatrix} & & \\ & 1 & \\ & & \end{pmatrix} \end{array} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}^{-1} = \lambda_i \lambda_j^{-1} \cdot E_{ij}.$$

Hence: $\mathrm{Lie}\, G = \mathrm{Lie}\, T \oplus \bigoplus_{i \neq j} k E_{ij}$.

We have $\mathfrak{X}(T) = \bigoplus_{i=1}^{n} \mathbb{Z} \varepsilon_i \Big/ \sum \varepsilon_i = 0$.

$$R = \{ \bar{\varepsilon}_i - \varepsilon_j \mid i \neq j \}.$$

$$\mathfrak{X}^\vee(T) = \{ \sum \lambda_i \varepsilon_i^* \mid \sum \lambda_i = 0 \}.$$

Root SL_2's are: $\begin{array}{c} \\ i \to \end{array}\!\!\begin{array}{c} j \\ \downarrow \\ \begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ c & 0 & d \end{pmatrix} \end{array}$, hence $R^\vee = \{ \varepsilon_i^* - \varepsilon_j^* \mid i \neq j \}$.

Example: Sp_{2n}.

Recall: $Sp_{2n} = \{ X \in \cancel{Mat}\, GL_{2n} \mid X^t \tilde{J} X = \tilde{J} \}$.

We calculate:

$$\mathrm{Lie}\, Sp_{2n} = \{ (id + \varepsilon Y) \mid (id + \varepsilon Y)^t \tilde{J} (id + \varepsilon Y) = \tilde{J} \}$$

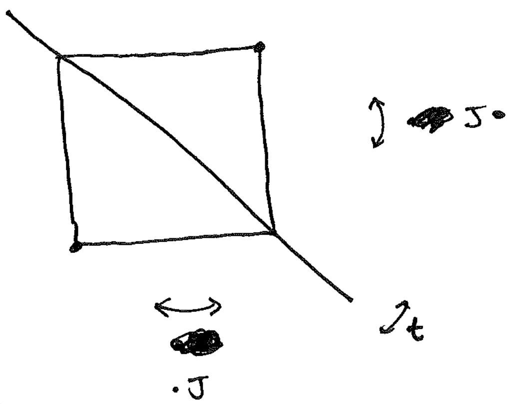
$$= \{ Y \mid Y^t \tilde{J} + \tilde{J} Y = 0 \}.$$

If we write $Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, relation becomes:

$$\begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} + \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0$$

$$\begin{pmatrix} -C^tJ & A^tJ \\ -D^tJ & B^tJ \end{pmatrix} + \begin{pmatrix} JC & JD \\ -JA & -JB \end{pmatrix} \iff \begin{array}{l} -C^tJ + JC = 0 \\ A^tJ + JD = 0 \\ -D^tJ - JA = 0 \\ B^tJ - JB = 0 \end{array}$$

$$\iff \begin{array}{l} C = JC^tJ = C^{\circledast} \\ D = -JA^tJ = -A^{\circledast} \\ B = JB^tJ. = B^{\circledast} \end{array}$$



Where $C \mapsto C^{\circledast}$ is "anti-diagonal transpose".

Hence: $$\mathrm{Lie}\, Sp_{2n} = \left\{ \begin{pmatrix} A & B \\ C & -A^{\circledast} \end{pmatrix} \middle| \begin{array}{l} C = C^{\circledast} \\ B = B^{\circledast} \end{array} \right\}.$$

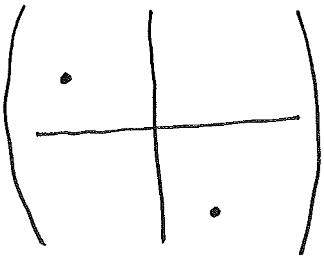
Fact: $$T = \mathrm{diag} \cap \cancel{GL_{2n}}\, Sp_{2n} = \left\{ \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_2^{-1} & \\ & & & & \lambda_1 \end{pmatrix} \middle| \lambda_1, \dots, \lambda_n \in k^\times \right\}$$

is a maximal torus in Sp_{2n}.

Weight spaces for T-action on ~~Sp~~ Lie Sp_{2n} are of three types ~~two classes~~:

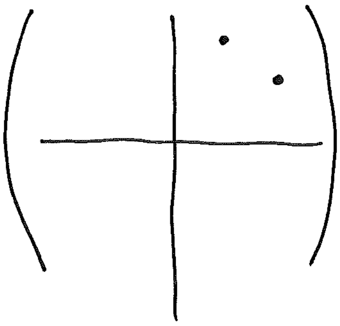
$$\mathcal{X}(T) = \bigoplus_{i=1}^{n} \mathbb{Z} e_i \; , \qquad e_i \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & \lambda_n^{-1} & & \\ & & & & \ddots & \\ & & & & & \lambda_1^{-1} \end{pmatrix} = \lambda_i .$$

$E_{ij} - E_{2n+1-i,2n+1-j}$, $1 \le i \le n$, $i \neq j$.

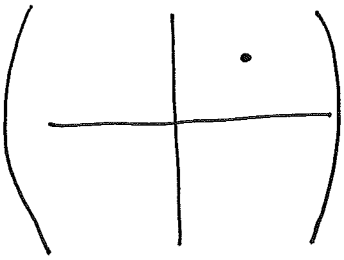


$\longrightarrow$ weight: $e_i - e_j$

$E_{i,2n+1-j} + E_{2n+1-i,j}$, $1 \le i, j \le n$ $\longrightarrow$ weight: $e_i + e_j$.
$i \neq j$.



$E_{i,2n-i}$, $1 \le i \le n$. $\longrightarrow$ weight $2e_i$.



Lie Sp_{2n} is the direct sum of these weight spaces, and hence these are all weights.

Hence we have determined $R \subset \mathcal{X}(T)$.

To determine the coroots we need to construct root SL_2's.

Root Sl_2's in Sp_4:

$$\left(\begin{array}{cc|cc} a & b & & \\ c & d & & \\ \hline & & a & -b \\ & & -c & d \end{array}\right) \qquad \left(\begin{array}{cc|cc} 1 & & & \\ & a & b & \\ \hline & c & d & \\ & & & 1 \end{array}\right) \qquad \left(\begin{array}{cc|cc} a & & & b \\ & 1 & 0 & \\ \hline & 0 & 1 & \\ c & & & d \end{array}\right)$$

$$\rightsquigarrow \text{coroot} \qquad\qquad \rightsquigarrow \text{coroot} \qquad\qquad \rightsquigarrow$$

$$e_1^* - e_2^* \qquad\qquad \varepsilon_2^* \qquad\qquad \varepsilon_1^* .$$

$$\left(\begin{array}{cc|cc} a & & b & \\ & a & & b \\ \hline c & & d & \\ & c & & d \end{array}\right) \quad \overset{\text{coroot}}{\rightsquigarrow} \quad \varepsilon_1^* + \varepsilon_2^* .$$

Hence: $R^\vee = \{\pm(\varepsilon_1^* - \varepsilon_2^*), \pm(\varepsilon_i), \pm(\varepsilon_1 + \varepsilon_2)\}$.

In general: $R^\vee = \{\varepsilon_i^* \pm \varepsilon_j^* \mid i \neq j\} \cup \{\varepsilon_i^*\}$.

<u>Exercise:</u> Determine the root data for SO_{2n+1}.

Show that SO_{2n+1} and Sp_{2n} are Langlands dual.

Comment on Borel subgroups here.