Suppose A, B are rings and ${}_A M_B$ is an (A,B) bimodule.

$\boxed{\text{Induction into induction!}}$

Lemma: "hom-tensor adjunction" (${}_A M_B \otimes (-)$ is left adjoint to $\mathrm{Hom}_A({}_A M_B, -)$).

i.e. $$\mathrm{Hom}_{A-}({}_A M_B \otimes X, Y) \xrightarrow{\sim} \mathrm{Hom}_{B-}(X, \mathrm{Hom}_A({}_A M_B, Y)).$$

Now suppose $B \subset A$ is a subring. Applying $\otimes$-hom adjunctions to the following bimodules:

$${}_A A_B : \quad \Big(\underbrace{A \otimes_B (-)}_{\text{"induction"}}, \quad \mathrm{Hom}_A(A, -) = \mathrm{res}^B_A(-) \Big)$$

$${}_B A_A : \quad \Big(\mathrm{res}^B_A(-) = A \otimes_A (-), \quad \underbrace{\mathrm{Hom}_{B-}(A, -)}_{\text{"coinduction"}} \Big).$$

Rmk: coinduction can be enormous (e.g. $\mathrm{coind}^k_A(k) = A^*$).

This might explain why induction shows up more often.

Example: Suppose $H < G$ are finite groups.

"Frobenius reciprocity"

induction: $kG \otimes_{kH} (-) =: \mathrm{ind}^G_H(\,)$

coinduction: $\mathrm{Hom}_{kH-}(kG, -) = \mathrm{Hom}_H(G, -)$.

" $\mathrm{coind}^G_H(V)$ = maps of H-sets $G \to V$ ".

Exercise: Show that one has an isomorphism of functors $\mathrm{coind}^G_H \xrightarrow{\sim} \mathrm{coind}^G_H$.

Aside (important)

Suppose $F: \mathcal{C} \to \mathcal{D}$ is a functor.

If F has a left adjoint ${}^{\vee}F$ (resp. right adjoint) $F^{\vee}$ then it is unique up to unique isomorphism. Hence this is a yes/no question!
(rare in category theory...)

However, if ${}^{\vee}F$ and $F^{\vee}$ exist we can ask: are they isomorphic?

Space of isomorphisms is a torsor over $\mathrm{Aut}({}^{\vee}F)$.

Hence if G is both a left and right adjoint of F then this is much more structure (very rich).

In our example, $\mathrm{Aut}(\mathrm{ind}_H^G) \cong \mathrm{Aut}({}_{kG}kG_{kH}) \cong (kG)^{\times}$.

Induction for algebraic groups

G group scheme / k.
H closed subgroup scheme.

$r: k[G] \longrightarrow k[H]$ restriction.

Recall that the restriction functor is given by

$V \in G\text{-mod}$: $V \longrightarrow V \otimes k[G] \rightsquigarrow V \longrightarrow V \otimes k[G] \longrightarrow V \otimes k[H]$ in H-mod.

Res_H^G

Lemma: "Frobenius reciprocity for group schemes"

Res_H^G has a right adjoint ind_H^G given by

$$\mathrm{ind}_H^G(V) = \mathrm{Hom}_{H\text{-varieties}}(G, V) = \mathrm{Hom}_{k[H]\text{-comodules}}(V, k[G])$$

$f: G \to V$ s.t. $f(hg) = h \cdot f(g) \ \forall h \in H$

Here we use the comodule structure induced by the left action on $H \circlearrowright G$.

(i.e. $k[G] \to k[G] \otimes k[G] \xrightarrow{S \otimes id} k[G] \otimes k[G] \xrightarrow{swap} k[G] \otimes k[G] \to k[G] \otimes k[H]$).

Silly example: $\mathrm{ind}_1^G k = k[G]$.

Exercise: Recall that if $\mathrm{char}\, k = 0$, $\mathrm{Dist}\, \mathbb{G}_a = k[\frac{\partial}{\partial z}]$

$\mathbb{G}_a$, $k[\mathbb{G}_a] = k[z]$.

Moreover,

$$\begin{tikzcd} \mathrm{Rep}\, \mathbb{G}_a \arrow[r] \arrow[dr, "\sim"'] & \mathrm{Dist}\, \mathbb{G}_a\text{-modules} \\ & \mathrm{Nilp} \arrow[u, hook] \end{tikzcd}$$

is fully-faithful with image those $\mathrm{Dist}\, \mathbb{G}_a$-modules where $\frac{\partial}{\partial z}$ acts locally ~~finitely~~ nilpotently.

① Show that $k[z]$ is an injective $k[\frac{\partial}{\partial z}]$-module.

② Show that Nilp does not contain any projective modules $\neq 0$.

③ Deduce that $\mathrm{Res}^{1}_{\mathbb{G}_a}$ does not have a left adjoint.

Rmk: Res^{H}_{G} rarely has a left adjoint.

Example: induction for SL_2.

$$SL_2 = \left\{\begin{pmatrix} x & y \\ z & w \end{pmatrix} \middle| \det = 1\right\} \supset \mathcal{U} = \left\{\begin{pmatrix} x & y \\ z & w \end{pmatrix} \middle| x \neq 0\right\}$$

Zariski open and dense.

Set $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, $B^- = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$.

$x \neq 0$, $\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & 0 \\ z & x^{-1} \end{pmatrix}\begin{pmatrix} 1 & x^{-1}y \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & a\lambda \\ b & b\lambda + a^{-1} \end{pmatrix}$.

Check: $B^- \times \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \xrightarrow[\sim]{\text{mult}} \mathcal{U}$.

Fix $\lambda \in \mathbb{Z}$:

Suppose k_λ is the B^--module: $\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \longmapsto \lambda\begin{pmatrix} a & b \\ b & a^{-1} \end{pmatrix} = a^\lambda$

Then $\mathrm{ind}_{B^-}^{G} k_\lambda = \{ f : SL_2 \longrightarrow k_\lambda \mid f(bg) = \lambda(b) f(g) \quad \forall b \in B \}$.

$$\cap$$

$$= \{ f : U \longrightarrow k_\lambda \mid f(bg) = \lambda(b) f(g) \ \forall b \in B \}.$$

$$\| $$

$$k[x^{-1}y].$$

Choose f in this latter space, then

$$f\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}\right) = f\left(\begin{pmatrix} x & 0 \\ z & x^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & x^{-1}y \\ 0 & 1 \end{pmatrix}\right) = x^\lambda f\left(\begin{pmatrix} 1 & x^{-1}y \\ 0 & 1 \end{pmatrix}\right).$$

Rmk: $B^- \backslash U \cong \mathbb{A}^1$.

~~Suppose f is a polynomial in $x^{-1}y$. Then~~

$\Longrightarrow$ f is determined by its restriction to $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$.

If $f\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right) = \sum a_i y^i$ then $f\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}\right) = x^\lambda \sum a_i (x^{-1}y)^i$.

extends to SL_2 $\iff$ $a_i = 0$ for $i > d$.

Thus: $\mathrm{ind}_{B^-}^{G} k_\lambda =$ spanned by $x^d, x^{d-1}y, \ldots, y^d$.

(zero if $d < 0$).

<u>Important observations:</u> ① $\mathrm{ind}_{B^-}^{G} k_\lambda$ is finite-dimensional.

("real reason": $B^- \backslash G \cong \mathbb{P}^1$ is projective).

② Again, we see functions on the group playing a key role.

② Let $V = (*, *)$ be the right SL_2-module of row vectors.

We have a map $SL_2 \longrightarrow V$ of right SL_2-varieties.

Induces: $k[x,y] \longrightarrow k[x,y,z,w]/\det$, and hence

$$\nabla_\lambda = \begin{matrix}\text{homogeneous} \\ \text{polys of degree} \\ \lambda \text{ in } x, y\end{matrix} = \operatorname{ind}_{B^-}^{G} k_\lambda .$$

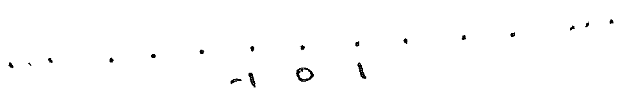
Rmk: In general one can interpret induction as follows: $V \in \operatorname{Rep} H$ $\rightsquigarrow$ G-equivariant vector bundle on $H \backslash G$ $\rightsquigarrow$ $\operatorname{ind}_H^G V$ = global sections.

<u>Chevalley's theorem:</u> Suppose G is a reductive algebraic group $/ k = \bar{k}$.

Fix a maximal torus $T \subset G$

Last time: $\rightsquigarrow$ $(\mathfrak{X}, R, \mathfrak{X}^\vee, R^\vee)$ Root datum.

<u>E.g.</u> ① $G = \mathbb{G}_m$, $\mathfrak{X} = \mathfrak{X}^\vee = \mathbb{Z}$, $R = R^\vee = \emptyset$.

$\mathfrak{X}$  $\mathfrak{X} = \mathfrak{X}_+$

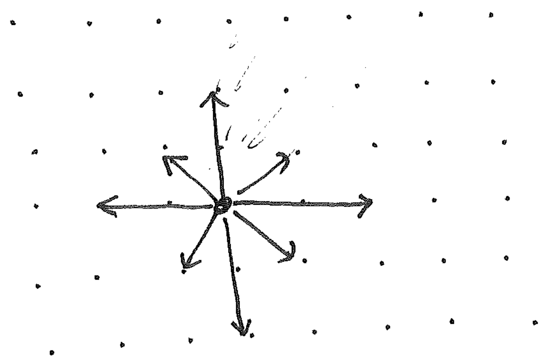
② $G = SL_2$, $\mathfrak{X} = \mathfrak{X}^\vee = \mathbb{Z}$, $R = \{\pm 2\}$, $R^\vee = \{\pm 1\}$.

$\mathfrak{X}$  $\mathfrak{X}_+ = \mathbb{Z}_{\geq 0}$

③ $G = Sp_4$: $\mathfrak{X} = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ $\quad R = \{\pm e_1 \pm e_2\} \cup \{\pm 2e_i\}$.

$\mathfrak{X}^\vee = \mathbb{Z}e_1^* \oplus \mathbb{Z}e_2^*$. $\quad R^\vee = \{\pm e_1^* \pm e_2^*\} \cup \{\pm 2e_i^*\}$.

$\mathcal{X}$



Recall: Borel subgroups containing $T \overset{\sim}{\longleftrightarrow}$ systems of positive roots $R^+ \subset R$.

$B \longleftrightarrow$ characters in $\mathrm{Lie}\, B \cap R$.

Fix a system of positive roots R_+ and let B^- be the Borel associated to the negative roots $-R_+$.

$$\mathcal{X}_+ = \{ \lambda \in \mathcal{X} \mid \langle \alpha^\vee, \lambda \rangle \geq 0 \quad \forall \alpha \in R_+ \}.$$

"dominant weights".

$B \overset{q}{\twoheadrightarrow} B/[B,B] \cong T$, k_λ a B module via $\lambda(b) = \lambda(q(b))$.

Chevalley's thm: (1) $\mathrm{ind}_{B^-}^G k_\lambda \neq 0 \Leftrightarrow \lambda \in \mathcal{X}_+$.

(2)
$$\left\{ \begin{matrix} \text{Irreducible} \\ \text{reps of } G \end{matrix} \right\} /_{\cong} \longleftrightarrow \mathcal{X}_+$$

$$\psi$$

$$L_\lambda := \mathrm{socle}\left(\mathrm{ind}_{B^-}^G k_\lambda \right) \longleftarrow\!\!\shortmid \lambda$$

Rmk: $\dim L_\lambda$ still unknown in general!

(subject of Lusztig's conjecture, ...)

We give the proof for SL_2 to illustrate ideas:

Set $U^- = R_u B^- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \cong \mathbb{G}_a$.

① $V \neq 0 \Rightarrow V^{U^-} \neq 0$ (only irreps. of $\mathbb{G}_a$ are trivial)
a $\mathbb{G}_a$-module

② $0 \neq V \in B^-$ mod $\Rightarrow$ $\exists \lambda$ s.t. $\mathrm{Hom}_{B^-}(V, k_\lambda) \neq 0$.
finite dim

(take projection onto a highest weight vector)

③ $\lambda \in \mathcal{X}_+ = \mathbb{Z}_{\geq 0} \Rightarrow \dim \left(\mathrm{ind}_{B^-}^G k_\lambda\right)^{U^-} = 1$.

indeed, $\mathrm{ind}_{B^-}^G k_\lambda = kx^\lambda \oplus kx^{\lambda-1}y \oplus \ldots \oplus ky^\lambda$.

$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \cdot x^\alpha y^\beta = x^\alpha (bx+y)^\beta = x^\alpha y^\beta \Leftrightarrow \beta = 0$.

(need to be a little more careful of course).

②+① $\Rightarrow$ socle$\left(\mathrm{ind}_{B^-}^G k_\lambda\right)$ is simple.

Now let L be simple, by ② there exists λ s.t.

$0 \neq \alpha \in \mathrm{Hom}_{B^-}(\mathrm{res}_{B^-}^G L, k_\lambda)$, $0 \neq \tilde{\alpha} : L \longrightarrow \mathrm{ind}_{B^-}^G k_\lambda$ is injective.

Hence $L = \mathrm{socle}\left(\mathrm{ind}_{B^-}^G k_\lambda\right)$. $\square$