A complement to "why reductive groups":

Exercise: Suppose G/k is an algebraic group and L is a simple G-module.

Then $R_u G$ (unipotent radical) acts trivially on L and so

L is in fact a simple $G/R_u G$-module.

Moral: to understand simple modules for ~~reductive~~ algebraic groups it is enough to understand simple modules for reductive groups!

Last time: $G/_{k=\bar{k}}$ a reductive algebraic group.

$$T \subset G \text{ maximal torus } \overset{\text{root}}{\underset{\text{datum}}{\rightsquigarrow}} (\mathcal{X}(T), R, \mathcal{X}^\vee(T), R^\vee).$$

$R^+ \subset R$ choice of positive roots, B corresponding Borel subgroup

B^- Borel cor. to $-R_+$.

$$\mathcal{X}_+ := \{\lambda \in \mathcal{X} \mid \langle \alpha^\vee, \lambda \rangle \geq 0, \text{ for } \alpha \in R_+\} \text{ dominant wts.}$$

Chevalley's thm: $\lambda \in \mathcal{X} \rightsquigarrow k_\lambda$ one-dimensional B-module.

① $\mathrm{ind}_{B_-}^G k_\lambda \neq 0 \iff \lambda \in \mathcal{X}_+$.

②
$$\begin{array}{ccc} \{\text{simple } G\text{-modules}\}/_{\cong} & \longleftrightarrow & \mathcal{X}_+ \\ \cup & & \Psi \\ L_\lambda := \mathrm{socle}\ \mathrm{ind}_{B^-}^G k_\lambda & \longleftarrow & \lambda \end{array}$$

For simplicity we assume that G is • semi-simple (i.e. $\mathbb{Q}R = \underbrace{\mathcal{X} \otimes \mathbb{Q}}_{\mathcal{X}_\mathbb{Q}}$)

• simply connected (i.e. $\{\lambda \in \mathcal{X}_\mathbb{Q} \mid \langle \alpha^\vee, \lambda \rangle \in \mathbb{Z}\ \forall \alpha\} = \mathcal{X}$).

E.g. • GL_n is not semi-simple, but SL_n is.

• SL_2, Sp_4 are simply connected, PGL_2 is not.

$\boxed{\text{DRAW PICTURES ON THE BOARD HERE}}$

Since our root system is assumed simply connected:

$$\mathbb{Z}R^\vee \times \mathfrak{X} \to \mathbb{Z} \text{ is a perfect pairing.}$$

Hence we can find ϖ_s s.t. $\langle \alpha_t^\vee, \varpi_s \rangle = \delta_{s,t}$.

The set $\{\varpi_s \mid s \in S\}$ is called ~~to~~ a <u>basis of fundamental weights.</u>

<u>Characteristic 0: Weyl dimension formula, Weyl character formula.</u>

Recall that in this case $\operatorname{Rep} G$ is semi-simple, if G is semi-simple $\operatorname{Rep} G \cong \operatorname{Rep}_{f.d.} \operatorname{Lie} G$

can be studied algebraically

<u>Exercise:</u> Define $\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$. Using that s permutes $R_+ \setminus \{\alpha\}$ and sends $\alpha \mapsto -\alpha$ show that $s(\rho) = \rho - \alpha_s$ $\forall s$. Deduce that

$$\rho = \sum_{s \in S} \varpi_s .$$

Write elements of $\mathbb{Z}[\mathfrak{X}]$ as $\sum_{\lambda \in \mathfrak{X}} m_\lambda e^\lambda$. Given $V \in \operatorname{Rep} G$ finite-dim. define

$$\operatorname{ch} V = \sum_{\lambda \in \mathfrak{X}} (\dim V_\lambda) e^\lambda . \qquad \text{"character".}$$

<u>Remark</u> (perhaps verbal) In the setting of a compact Lie group, K. Hence $K = \bigcup_{g \in K} g T_c g^{-1}$, where T_c denotes a maximal torus. Hence to know the trace of any $g \in K$ it is enough to know the trace of any T_c. This is determined by the above expression.

<u>Example:</u> $G = SL_2$. Because $\operatorname{char} k = 0$ each $\nabla_n = k[X,Y]_{\text{homog. of degree } n}$ is simple.

$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \cdot X = \lambda X$, $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \cdot Y = \lambda^{-1} Y$. Hence

$$\operatorname{ch} \nabla_0 = e^0$$

$$\operatorname{ch} \nabla_1 = e^{-\varpi_1} + e^{\varpi} \qquad kY \oplus kX$$

$$\operatorname{ch} \nabla_2 = e^{-2\varpi} + e^0 + e^{2\varpi}$$

$$\vdots$$

<u>Example:</u> $\mathfrak{g} = \mathrm{Lie}\ SL_3$ with adjoint representation. $M = 3\times 3$-matrices.

$M \cong sl_3 \oplus \mathrm{triv}.$

$$\mathcal{X}^*(T) = \mathbb{Z}\{e_1, e_2, e_3\}/(e_1+e_2+e_3).$$

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<u>Weights:</u>

$$\begin{pmatrix} 0 & e_1-e_2 & e_1-e_3 \\ -e_2+e_3 & 0 & e_2-e_3 \\ -e_1+e_3 & -e_1+e_2 & 0 \end{pmatrix}$$

$\alpha_1 = e_1 - e_2$

$\alpha_2 = e_2 - e_3.$

$$\mathrm{ch}\ M = e^{-\alpha_1-\alpha_2} + e^{-\alpha_1} + e^{-\alpha_2} + 3e^0 + e^{\alpha_1} + e^{\alpha_2} + e^{\alpha_1+\alpha_2}.$$

<u>Hence:</u> $$\mathrm{ch}\ sl_3 = e^{-\alpha_1-\alpha_2} + e^{-\alpha_1} + e^{-\alpha_2} + 2e^0 + e^{\alpha_1} + e^{\alpha_2} + e^{\alpha_1+\alpha_2}.$$

$W = \langle s \mid s \in S \rangle \subset \mathrm{Aut}(\mathcal{X})$ "Weyl group".

(W, S) is a Coxeter system, i.e. if $m_{st} := \mathrm{order}(st)$ then

$$W = \langle s \in S \mid s^2 = id,\ (st)^{m_{st}} = 1 \rangle.$$

$\ell : W \to \mathbb{Z}$ $\ell(w) =$ length of an expression for w of minimal length in S

$$\ell(w) = \#(R_+ \cap w(R_+)).$$

Weyl character formula:

$$\mathrm{ch}\ L_\lambda = \frac{\sum_{x\in W} (-1)^{\ell(x)} e^{x\cdot\lambda}}{\sum_{x\in W} (-1)^{\ell(x)} e^{x\cdot 0}}$$

<u>Example:</u> For Sl_2,

$$\mathrm{ch}\ \nabla_n = \frac{e^n - e^{s\cdot n}}{e^0 - e^{s\cdot 0}} = \frac{e^n - e^{-n-2}}{1 - e^{-2}} = \frac{e^{n+1} - e^{-n-1}}{e - e^{-1}} = e^n + e^{n-2} + \dots + e^{-n}.$$

Weyl dimension formula:

$$\dim L_\lambda = \frac{\prod_{\alpha\in R^\vee} \langle \alpha^\vee, \lambda+\rho\rangle}{\prod_{\alpha\in R^\vee} \langle \alpha^\vee, \rho\rangle}$$

Eg: For Sl_2, gives

$$\dim L_n = \frac{\langle \alpha^\vee, n\varpi+\varpi\rangle}{\langle \alpha^\vee, \varpi\rangle} = \frac{n+1}{1} = n+1$$

Now switch to char $k = p > 0$.

Basic problem (still unsolved in general): determine $\mathrm{ch}\, L_\lambda$ for all $\lambda \in \mathcal{X}_+$.

Goal for the rest of the lecture: explain what was known up to 1979.

For $\lambda \in \mathcal{X}_+$ set $\nabla_\lambda := \mathrm{ind}_{B^-}^{G} k_\lambda$.

There exists a unique automorphism Chevalley : $G \to G$ s.t. acts on root system by $-w_0$, where w_0 is the longest element in W (sends $R_+ \to -R_+$).

Define $\Delta_\lambda = ((\nabla_\lambda)^*)^{\text{Chevalley}}$. "Weyl module".

Thm (consequence of "Kempf vanishing"): $\mathrm{ch}\, \nabla_\lambda = \mathrm{ch}\, \Delta_\lambda = \chi_\lambda$.

Revised basic problem: find expressions $\mathrm{ch}\, L_\lambda = \sum m_{\mu\lambda} \chi_\mu$.

STEINBERG TENSOR PRODUCT THEOREM

Calculations suggest that there is an affine Weyl group in the story!

DRAW PICTURES FOR Sl_2, Sp_4

$\cong W \ltimes \mathbb{Z}R$

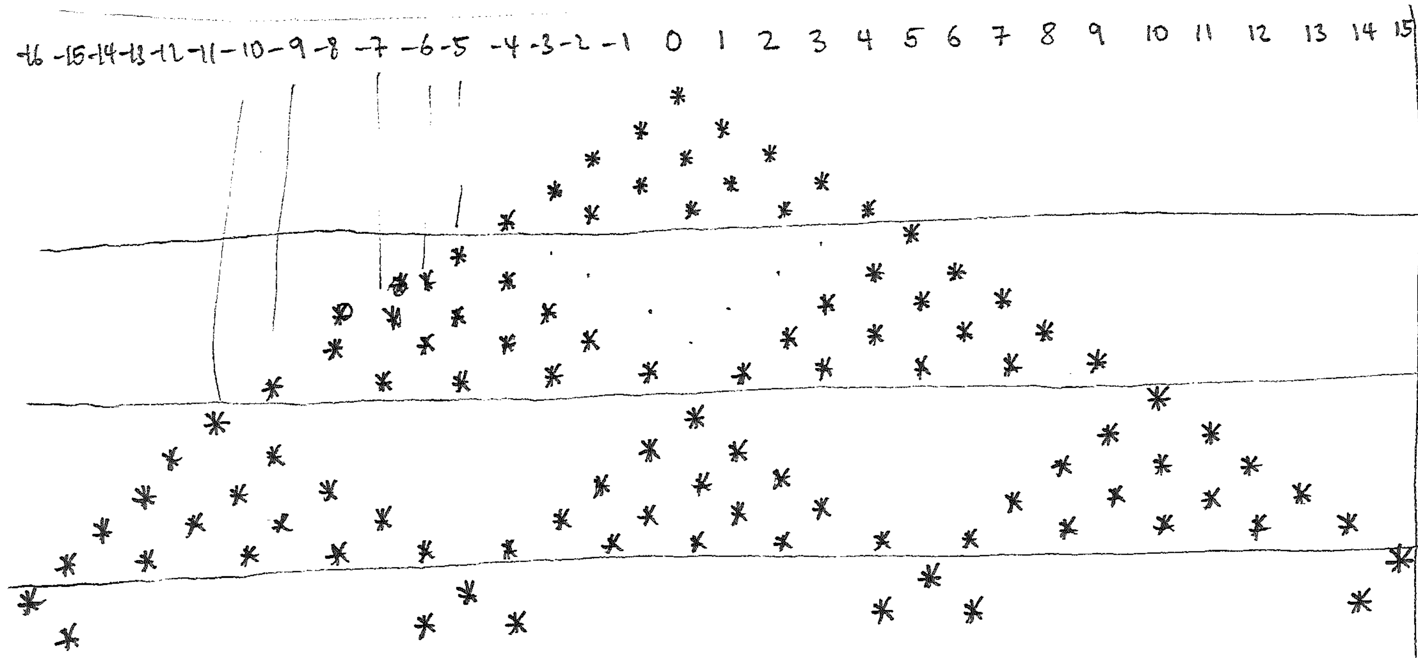
Recall: $\mathcal{X}_{\mathbb{R}} := \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{R}$.

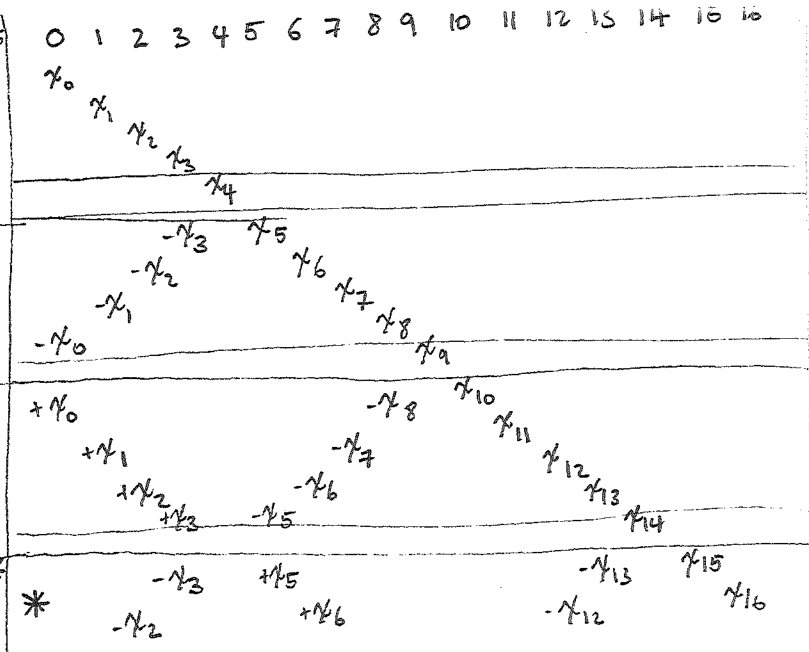
The group $W_{\text{aff}} := \langle W, \mathbb{Z}R \rangle \subset \mathrm{Aff}(\mathcal{X}_{\mathbb{R}})$ is generated by affine reflections "affine reflection group".

All affine reflection groups are affine Weyl groups associated to root systems.

We need a "p-dilated dot" version.

$\cong W_{\text{aff},p}^{\bullet} = \langle W\bullet, p\mathbb{Z}R \rangle \subset \mathrm{Aff}(\mathcal{X}_{\mathbb{R}})$.





Sl_2, $p=5$

This pattern persists until

$$\mathrm{ch}\, L_{25} = \chi_{25} - \chi_{23}$$

"p^2-effects".

If one did $p = \cancel{5}\, 7, 11, 13$ one would get an identical picture but "stretched" out" by p.

We have $\mathcal{X}_+ = \bigoplus \mathbb{Z}_{\geq 0} \cdot \varpi_s \subset \mathcal{X}$.

Hence any λ can be written

$$\lambda = \lambda_0 + p\lambda_1 + p^2\lambda_2 + \dots + p^m\lambda_m \quad \text{with } \lambda_i \in \mathcal{X}_{<p} = \{\lambda \in \mathcal{X}_+ \mid \langle \alpha^\vee, \lambda \rangle \lneq p\}$$

$$= \{\textstyle\sum \gamma_s \varpi_s \mid 0 \leq \gamma_s < p \ \forall s\}$$

<u>Theorem</u> (Steinberg Tensor Product Theorem)

With λ as above:

$$L_\lambda \cong L_{\lambda_0} \otimes L_{\lambda_1}^{Fr} \otimes L_{\lambda_2}^{(Fr)^2} \otimes \dots \otimes L_{\lambda_m}^{(Fr)^m}.$$

This reduces the problem of determining $\mathrm{ch}\, L_\lambda$ to the problem of for $\lambda \in \mathcal{X}_{<p}$. This is a <u>finite</u> problem.

<u>Ex:</u> For SL_2, $\mathcal{X}_{<p} = \{0, 1, \dots, p-1\}$ hence St⊗Thm completely solves the problem in this case.

Another remarkable theorem is Steinberg restriction theorem:

<u>Theorem:</u> The restriction functor, (Suppose $G/\mathbb{F}_p$ is split reductive and $q = p^m$.)

$$\mathrm{Rep}\, G \longrightarrow \mathrm{Rep}\, G(\mathbb{F}_q)$$

induces a bijection

$$\mathrm{Irr}\, kG(\mathbb{F}_p) \xleftrightarrow{\sim} \mathcal{X}_{<q}.$$

$$\mathrm{res}_G^{G(\mathbb{F}_p)} L_\lambda \longleftrightarrow \lambda$$

A fundamental domain for the action of $W_{aff,p}$ is

$$C^p_{-,\mathbb{R}} := \{ \lambda \in \mathfrak{X}_{\mathbb{R}} \mid -p \leq \langle \alpha^\vee, \lambda+\rho \rangle \leq 0 \quad \forall \alpha \in R_+ \}.$$

$$C^p_- := C^p_{-,\mathbb{R}} \cap \mathfrak{X}.$$

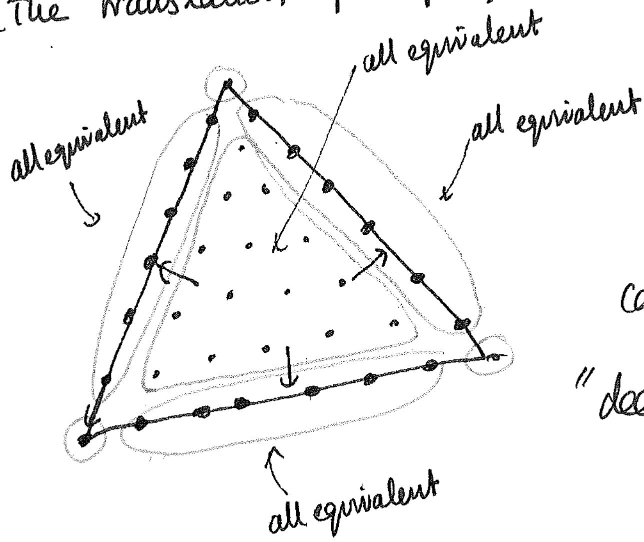
For $\mu \in C^p_-$ we define:

$$\mathrm{Rep}_\mu = \langle L_\lambda \mid \lambda \in W_{aff,p} \cdot \mu \rangle.$$

Theorem: (The linkage principle) (Verma, Humphreys, Andersen, Jantzen)

$$\mathrm{Rep}\, G = \bigoplus_{\mu \in C^p_-} \mathrm{Rep}_\mu.$$

Theorem: (The translation principle) (Jantzen)



can carry information "deeper into facets".

If the interior of $C^{p}_{-,\mathbb{R}}$ contains an element of $\mathcal{X}$ then we can reduce to this block.

<u>Lemma:</u> $\operatorname{int} C^{p}_{-,\mathbb{R}} \cap \mathcal{X} \neq \emptyset \iff p \geq h$ = Coxeter number $= \langle \alpha_0^\vee, \rho \rangle + 1$

↑ highest short root.

$\mathrm{Rep}_0 := \mathrm{Rep}_{-2\rho}$ "principal block" (block of trivial representation).

<u>Lusztig conjecture:</u> if $p \geq h$ and "no p^2 effects" then

$$\operatorname{ch} L_{x \cdot_p 0} = \sum_{\substack{y \leq x \\ y \cdot_p 0 \in \mathcal{X}^+}} \varepsilon_{yx} P_{y,x}(1) \operatorname{ch} \Delta_{y \cdot 0}$$