

Recall from last time: k field

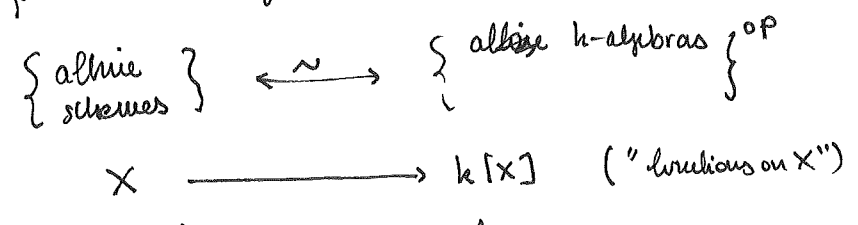
$k = \bar{k}$:

$X \subset k^n$ algebraic subset
 $\xrightarrow{\text{Nullstellensatz}}$ $k[X]$ regular
 functions on X

This leads us to consider all k -algebras (Grothendieck), leads to the world of schemes:

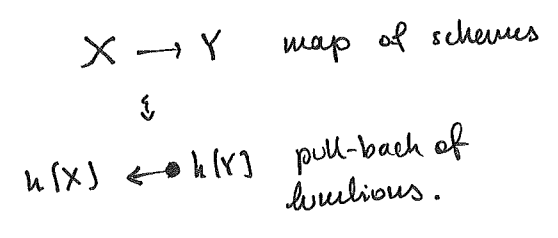
Equivalence of categories:

(G/A)



Throughout this course:
 algebraic scheme = algebraic k -scheme

$(\text{Spec } A, \mathcal{O}_{\text{Spec } A}) \longleftarrow A$
 Remember why we have to reverse the arrows:



Remark: One can think of the passage (G/A) as classical / quantum.
 in classical mechanics one has points, in quantum mechanics observables.

Why prime ideals? Historical motivation:

Consider 3 problems: $A = \mathbb{Q}[X] / (a_n X^n + \dots + a_0)$ ①, $A = \mathbb{R}[X, Y] / (X^2 + Y^2 - 1)$ ②, $A = \mathbb{Q}[X, Y, Z] / (X^n + Y^n - Z^n)$ ③ $n \geq 3$

Observation: Given any field k , $\text{Hom}(A, k)$ is a "solution in k ".

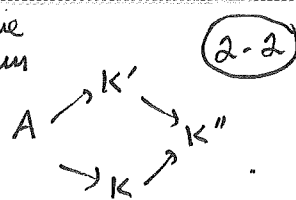
- ① Galois theory
- ② real algebraic geometry
- ③ Fermat's last theorem.

In all these problems, varying the field is essential.

Why not allow all fields?
 $k[X] \rightarrow K$ for

Definition: A geometric point is a homomorphism $\mathbb{A}^1 \rightarrow K$ for some k -field K .

Two geometric points are equivalent if there exists a commutative diagram with values in K, K'

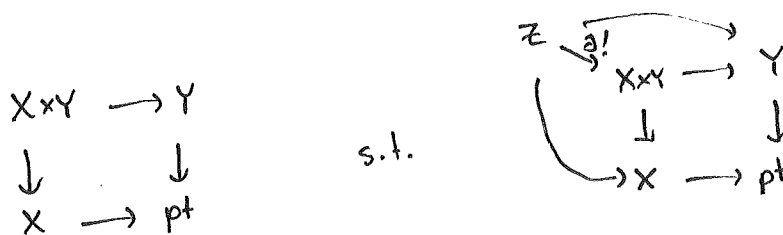


Exercise: $\{\text{geometric points}\} / \text{equivalence} \xrightarrow{\sim} \text{points of Spec } A$
(i.e. prime ideals).

Notation: $\text{pt} = \text{Spec } k$.

Lemma: The category of affine schemes has products.

(i.e. for affine schemes X, Y there exists a set diagram



Proof: Applying (G/A) (i.e. homing arrows around) we arrive at the axioms for a tensor product of k -algebras.
(i.e. $k[X \times Y] = k[X] \otimes_k k[Y]$). \square

Definition An affine group scheme is a group object in the category of affine schemes. That is, it is a scheme G together with maps:

$$m: G \times G \longrightarrow G \qquad u: \text{pt} \longrightarrow G \qquad i: G \longrightarrow G$$

such that the following diagrams commute:

What is a group scheme under (G/A) ?

(2-3)

Standard recipe: express group laws as diagrams, then reverse arrows.

Rule: One has to get used to reversing the arrows in this subject!

$$m: G \times G \longrightarrow G$$

"multiplication"

$$u: \{1\} \hookrightarrow G$$

"inclusion of unit"

$$i: G \longrightarrow G^{\#}$$

"inverse"

$$k[G] \xrightarrow{\Delta} k[G] \otimes k[G]$$

"comultiplication"

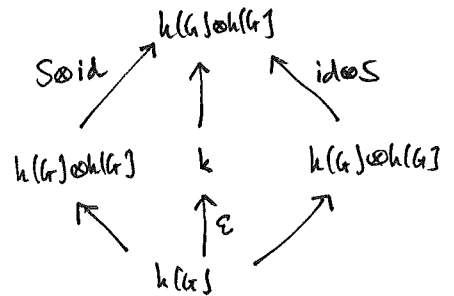
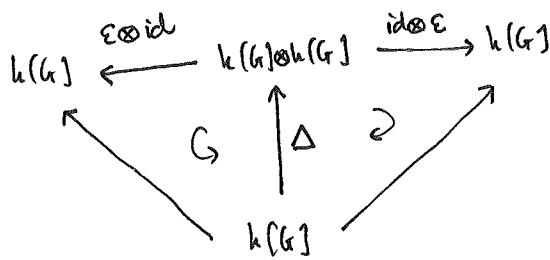
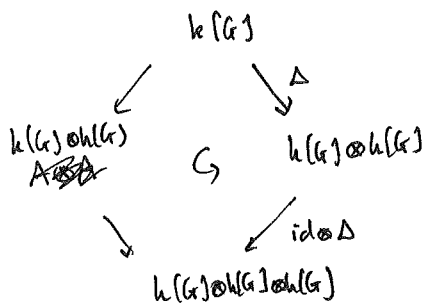
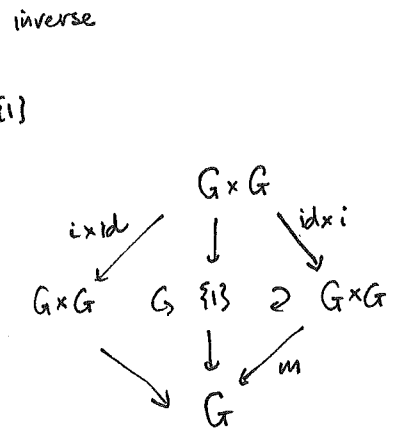
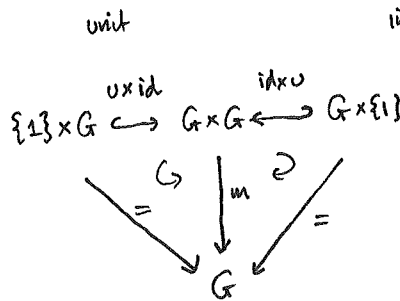
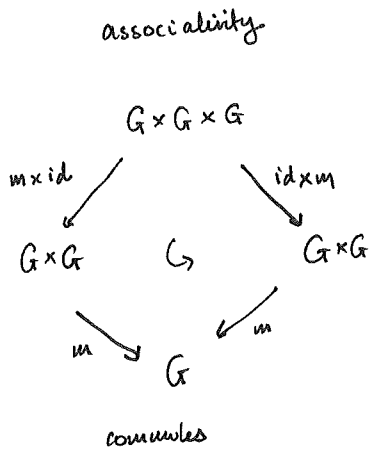
$$k \xrightarrow{\epsilon} k[G]$$

"counit"

$$k[G] \xrightarrow{S} k[G]$$

"antipode"

satisfying:



Exercise: Show that

w/ maps Δ, S, ϵ

- Any ring satisfying the diagrams above is called a Hopf algebra.

"space is to algebra as group is to Hopf algebras".

Hopf algebras are very special.

Exercise: Show that S is unique if it exists! (Uniqueness of inverses.)

What "is" comultiplication?

$$G \times G \longrightarrow G$$

comultiplication tells us that any coordinate on G is algebraic in functions on $G \times G$.

Under this equivalence a closed homomorphism of groups, homomorphisms of Hopf algebras $H \rightarrow G$ is a closed embedding if $k[G] \rightarrow k[H]$ is surjective. This is comultiplication. (2-4)

Examples:

G_a :
"additive group"

$$k[G_a] = k[x].$$

Hence comultiplication:

counit:

antipode:

Addition is

$$(x, y) \mapsto x + y.$$

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

$$\varepsilon(x) = 0.$$

$$S(x) = -x.$$

"multiplicative group":
 G_m

$$k[G_m] = k[x, x^{-1}]$$

counit:

antipode:

$$(x, y) \mapsto xy.$$

$$\Delta(x) = x \otimes x = (x \otimes 1) \cdot (1 \otimes x).$$

$$\varepsilon(x) = 1.$$

$$S(x) = x^{-1}.$$

" n th roots of unity": μ_n

$$k[\mu_n] = k[x] / (x^n - 1).$$

Same multiplication, comultiplication as above.

We have a closed embedding $\mu_n \hookrightarrow G_m$ corresponding to $k[x, x^{-1}] \rightarrow k[x] / (x^n - 1)$.

$$k = \mathbb{C}: k[\mu_n] = \mathbb{C} \times \mathbb{C} \times \dots \times \mathbb{C}, \text{ and } \text{Spec } \mathbb{C} = \dots \cong \mathbb{A}^1 / \mu_n.$$

Note: A very interesting thing happens when $k = \mathbb{F}_p$ and $n = p$.

$$k[\mu_p] = k[x] / (x^p - 1) = k[x] / (x - 1)^p \text{ which is } \underline{\text{non-reduced}}.$$

Thus non-reduced schemes appear naturally in this theory.

$\text{Spec } k[\mu_p] = \bullet$ "infinitesimal group scheme".

$SL_2: k[SL_2] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad-bc=1 \right\} = k(a,b,c,d)/(ad-bc-1)$. (2-5)

~~$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ a_2 c_1 + d_1 c_2 & a_2 d_1 + d_1 d_2 \end{pmatrix}$$~~

Hence:

$$\Delta(a) = a \otimes a + b \otimes c$$

$$\Delta(b) = a \otimes b + b \otimes d$$

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} x'_{11} & x'_{12} \\ x'_{21} & x'_{22} \end{pmatrix} = \begin{pmatrix} x_{11}x'_{11} + x_{12}x'_{21} & x_{11}x'_{12} + x_{12}x'_{22} \\ x_{21}x'_{11} + x_{22}x'_{21} & x_{21}x'_{12} + x_{22}x'_{22} \end{pmatrix}$$

$$\Delta(x_{ij}) = \sum_{1 \leq k \leq 2} x_{ik} \otimes x_{kj}$$

(Same formula works for GL_n .)

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}^{-1} = \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix}$$

hence:

$$S(x_{11}) = x_{22}$$

$$S(x_{12}) = -x_{21}$$

$$S(x_{21}) = -x_{12}$$

$$S(x_{22}) = x_{11}$$

DON'T FORGET
FINITE EXAMPLE
ON NEXT PAGE

$$\varepsilon(x_{ij}) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise.} \end{cases}$$

Functor of points: here we will formalize the "families notion from last lecture".

- Motivation:
- ① Grothendieck philosophy / Yoneda lemma.
 - ② Often in group theory we would like to agree with elements!

Yoneda Lemma: \mathcal{C} is a category, the functors $\text{Hom}(X, -): \mathcal{C} \rightarrow \text{Fun}(\mathcal{C}, \text{Sets})^{\text{op}}$
 $\text{Hom}(-, X): \mathcal{C} \rightarrow \text{Fun}(\mathcal{C}, \text{Sets})$
 are fully-faithful.

Philosophically: "X is determined in \mathcal{C} by the world in which it lives."

Suppose Γ is a ~~finite~~ group. If we apply Hom sets $(-, k)$ to the diagrams:

$$\Gamma \times \Gamma \rightarrow \Gamma$$

$$\{1\} \hookrightarrow \Gamma$$

$$\Gamma \xrightarrow{x \mapsto x^i} \Gamma$$

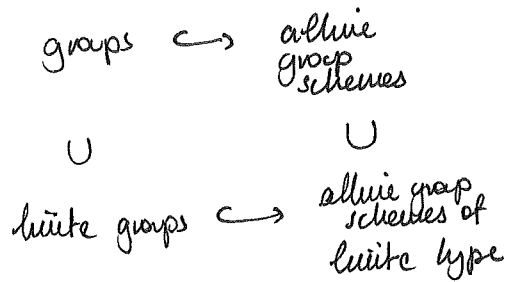
(2-5.5)

we obtain a k -algebra $k[\Gamma] = \text{functions } \Gamma \rightarrow k$ which we view as

a k -algebra under pointwise multiplication.

($k[\Gamma]$ is of finite type $\Leftrightarrow \Gamma$ is finite).

in this way one can embed:



Let G be a k -group scheme, and let A be a k -algebra. Set

2-6

$$G(A) := \text{Hom}_{k\text{-alg}}(k[G], A).$$

all the
X on k -scheme

Not one geometric object
but many geometric objects.

This set is called the A -points of G .

Examples: $G_a(A) = \text{Hom}_k(k[X], A) = A.$

$$G_m(A) = \text{Hom}(k[X, X^{-1}], A) = A^\times.$$

$$M_n(A) = \text{Hom}(k[X]/(X^n - 1), A) = \{a \in A \mid a^n = 1\}.$$

$$SL_2(A) = \text{Hom}(-, A) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A^4 \mid ad - bc = 1 \right\}.$$

Exercise: Check carefully that $G(A)$ is always a group via Δ, S, ε .

This allows us to suddenly to make sense of concepts in group theory,

we simply state the group theory definition but require it for all A .

Exercise: If $H \hookrightarrow G$ is a closed embedding then $H(A) \hookrightarrow G(A)$ is an

Example: ① $N < G$ is a normal subgroup scheme if $N(A) < G(A)$

is normal for all A .

Exercise: Find a group scheme G s.t.

② G is ~~commutative~~ ^{abelian} if $G(A)$ is ~~commutative~~ ^{abelian} for all A . $G(k)$ is abelian, but G is not!

② Consider the map $\text{Fr}: x \mapsto x^p$ on G_a in characteristic p . Is this injective??

Note that for any field k' , $(k' \xrightarrow{x \mapsto x^p} k')$

is ~~injective~~. $x^p = (x')^p$

$$\Rightarrow (x - x')^p = 0 \Rightarrow x = x'.$$

Hence however, Fr is not injective on arbitrary k -algebras.

In fact, Fr has a kernel $\{\alpha_p\}$ defined as follows:

$$k[\alpha_p] = k[X]/(X^p), \quad \Delta(x) = x \otimes 1 + 1 \otimes x \quad \varepsilon(x) = 0.$$

$S(x) = -x$. Again this is infinitesimal.