Recall from last time: k field

$k = \bar{k}$:

$$X \subset k^n \quad \overset{\text{Nullstellensatz}}{\leftrightsquigarrow} \quad k[X] \text{ regular}$$

algebraic subset — functions on X

This led us to consider **all** k-algebras (Grothendieck), leads to the world of schemes:

Equivalence of categories:

> Throughout this course: affine scheme = affine k-scheme

G/A
$$\{\text{affine schemes}\} \xleftrightarrow{\sim} \{\text{affine } k\text{-algebras}\}^{op}$$

$$X \longrightarrow k[X] \quad (\text{"functions on } X\text{"})$$

$$(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \longleftarrow A$$

Remember why we have to reverse the arrows: $X \to Y$ map of schemes

$$\Updownarrow$$

$k[X] \leftarrow k[Y]$ pull-back of functions.

Remark: One can think of the passage G/A as classical / quantum.

in classical mechanics one has points, in quantum mechanics observables.

Why prime ideals? Historical motivation:

Consider 3 problems: ① $A = \mathbb{Q}[X]/(a_nX^n + \ldots + a_0)$, ② $A = \mathbb{R}[X,Y]/(X^2+Y^2-1)$, ③ $n \geq 3$, $A = \mathbb{Q}[X,Y,Z]/(X^n+Y^n-Z^n)$.

Observation: Given any field k, $\operatorname{Hom}(A,k)$ is a "solution in k".

① Galois theory ② real algebraic geometry ③ Fermat's last theorem.

In all these problems, varying the field is essential. Why not allow all fields?

Definition: A geometric point is a homomorphism $k[X] \to K$ for some k-field K.

Two geometric points are equivalent if there exists a commutative diagram

$$\begin{tikzcd} & K' \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & K'' \\ & K \arrow[ur] & \end{tikzcd}$$

with values in K, K'.

<u>Exercise:</u> $\{\text{geometric points}\}/_{\text{equivalence}} \xleftrightarrow{\sim}$ points of Spec A (i.e. prime ideals).

Notation: $pt = \operatorname{Spec} k$.

<u>Lemma:</u> The category of affine schemes has products.
(i.e. for affine schemes X, Y there exists a ~~sche~~ diagram

$$\begin{tikzcd} X\times Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & pt \end{tikzcd} \quad \text{s.t.} \quad \begin{tikzcd} Z \arrow[dr, "\exists !"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X\times Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & X \arrow[r] & pt \end{tikzcd}$$

<u>Proof:</u> Applying (G/A) (i.e. turning arrows around) we arrive at the axioms for a tensor product of k-algebras.
(i.e. $k[X\times Y] = k[X] \otimes_k k[Y]$). □

<u>Definition</u> An <u>affine group scheme</u> is a group object in the category of affine schemes. That is, it is a scheme G together with maps:

$m: G\times G \longrightarrow G \qquad u: pt \longrightarrow G \qquad i: G \longrightarrow G$

such that the following diagrams commute:

What is a group scheme under (G/A) ?

Standard recipe: express group laws as diagrams, then reverse arrows.
Rmk: One has to get used to reversing the arrows in this subject!

$m : G\times G \longrightarrow G$ "multiplication"

$u : \{1\} \hookrightarrow G$ "inclusion of unit"

$i : G \longrightarrow G$ "inverse"

$k[G] \xrightarrow{\Delta} k[G]\otimes k[G]$ "comultiplication"

$k \xrightarrow{\varepsilon} k[G]$ "counit"

$k[G] \xrightarrow{S} k[G]$ "antipode"

satisfying:

associativity

$$\begin{tikzcd}
& G\times G\times G \arrow[dl, "m\times id"'] \arrow[dr, "id\times m"] & \\
G\times G \arrow[dr, "m"'] & \circlearrowleft & G\times G \arrow[dl, "m"] \\
& G &
\end{tikzcd}$$

commutes

unit

$$\begin{tikzcd}
\{1\}\times G \arrow[r, hook, "u\times id"] \arrow[dr, "="'] & G\times G \arrow[d, "m"] & G\times\{1\} \arrow[l, hook', "id\times u"'] \arrow[dl, "="] \\
& G &
\end{tikzcd}$$

inverse

$$\begin{tikzcd}
& G\times G \arrow[dl, "i\times id"'] \arrow[d] \arrow[dr, "id\times i"] & \\
G\times G \arrow[dr] & \{1\} \arrow[d] & G\times G \arrow[dl, "m"] \\
& G &
\end{tikzcd}$$

$$\begin{tikzcd}
& k[G] \arrow[dl] \arrow[dr, "\Delta"] & \\
k[G]\otimes k[G] \arrow[dr] & \circlearrowleft & k[G]\otimes k[G] \arrow[dl, "id\otimes\Delta"] \\
& k[G]\otimes k[G]\otimes k[G] &
\end{tikzcd}$$

$$\begin{tikzcd}
k[G] & k[G]\otimes k[G] \arrow[l, "\varepsilon\otimes id"'] \arrow[r, "id\otimes\varepsilon"] & k[G] \\
& k[G] \arrow[ul] \arrow[u, "\Delta"] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
& k[G]\otimes k[G] & \\
k[G]\otimes k[G] \arrow[ur, "S\otimes id"] & k \arrow[u] & k[G]\otimes k[G] \arrow[ul, "id\otimes S"'] \\
& k[G] \arrow[ul] \arrow[u, "\varepsilon"] \arrow[ur] &
\end{tikzcd}$$

~~Exercise: Show that~~

- Any ring w/ maps Δ, S, ε satisfying the diagrams above is called a Hopf algebra.

"space is to algebra as group is to Hopf algebras". Hopf algebras are very special.

Exercise: Show that S is unique if it exists! (Uniqueness of inverses.)

What "is" comultiplication?

$$G \times G \longrightarrow G$$

multiplication tells us that any coordinate on G is algebraic in functions on $G \times G$.

This is comultiplication.

* Under this equivalence a homomorphism of groups, homomorphisms of groups correspond to homomorphisms of Hopf algebras. $H \to G$ is a closed embedding if $k[G] \to k[H]$ is surjective.

Examples:

$\mathbb{G}_a$: "additive group"

$k[\mathbb{G}_a] = k[x]$.

Addition is $(x,y) \mapsto x+y$.

Hence comultiplication: $\Delta(X) = X \otimes 1 + 1 \otimes X$

counit: $\varepsilon(X) = 0$.

antipode: $S(X) = -X$.

"multiplicative group": $\mathbb{G}_m$

$k[\mathbb{G}_m] = k[X, X^{-1}]$

$(x,y) \mapsto xy$.

$\Delta(X) = X \otimes X = (X \otimes 1) \cdot (1 \otimes X)$.

counit: $\varepsilon(X) = 1$.

antipode: $S(X) = X^{-1}$.

"n^{th} roots of unity": μ_n $\quad k[\mu_n] = k[X]/(X^n - 1)$. Same multiplication, comultiplication as above.

We have a closed embedding $\mu_n \hookrightarrow \mathbb{G}_m$ corresponding to $k[X, X^{-1}] \twoheadrightarrow k[X]/(X^n-1)$.

$k = \mathbb{C}$: $\mathbb{C}[\mu_n] = \mathbb{C} \times \mathbb{C} \times \dots \times \mathbb{C}$, and $\operatorname{Spec} \mathbb{C} = \vdots \cong \mathbb{Z}/n\mathbb{Z}$.

Note: A very interesting thing happens when $k = \mathbb{F}_p$ and $n = p$.

$$k[\mu_p] = k[X]/(X^p - 1) = k[X]/(X-1)^p \quad \text{which is non-reduced.}$$

Thus non-reduced schemes appear naturally in this theory.

$\operatorname{Spec} k[\mu_p] = \bullet$ "infinitesimal group scheme".

$\underline{SL_2:}$ $\quad k[SL_2] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad-bc=1 \right\} = k[a,b,c,d]/(ad-bc-1).$

$$\cancel{\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}}$$

~~Hence:~~ $\cancel{\Delta(a) = a\otimes a + b\otimes c}$

$\cancel{\Delta(b) = a\otimes b + b\otimes d}$

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} x'_{11} & x'_{12} \\ x'_{21} & x'_{22} \end{pmatrix} = \begin{pmatrix} x_{11}x'_{11} + x_{12}x'_{21} & x_{11}x'_{12} + x_{12}x'_{22} \\ x_{21}x'_{11} + x_{22}x'_{21} & x_{21}x'_{12} + x_{22}x'_{22} \end{pmatrix}.$$

$$\Delta(x_{ij}) = \sum_{1\le k\le 2} x_{ik}\otimes x_{kj}.$$

(Same formula works for GL_n.)

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}^{-1} = \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{22} \end{pmatrix} \quad \text{hence:}$$

$S(x_{11}) = x_{22}$
$S(x_{12}) = -x_{12}$
$S(x_{21}) = -x_{21}$
$S(x_{22}) = x_{22}$.

$$\varepsilon(x_{ij}) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise.} \end{cases}$$

DON'T FORGET FINITE EXAMPLE ON NEXT PAGE

<u>Functor of points:</u> here we will formalize the "families notion from last lecture".

<u>Motivation:</u> ① Grothendiecks philosophy / Yoneda lemma.

② Often in group theory we would like to argue with elements!

<u>Yoneda Lemma:</u> $\mathcal{C}$ is a category, the functors $Hom(X,-): \mathcal{C} \to Fun(\mathcal{C}, Sets)^{op}$

$Hom(-,X): \mathcal{C} \to Fun(\mathcal{C}, Sets)$

are fully-faithful.

Philosophically: "X is determined ~~in~~ by the world in which it lives."

Suppose Γ is a ~~finite~~ group. If we apply $\mathrm{Hom}_{\mathrm{Sets}}(-, k)$ to the diagrams: (2-5.5)

$$\Gamma \times \Gamma \longrightarrow \Gamma \qquad \{1\} \hookrightarrow \Gamma \qquad \Gamma \xrightarrow[x \mapsto x^{-1}]{} \Gamma$$

we obtain a k-algebra $k[\Gamma]$ = functions $\Gamma \to k$ which we view as a k-algebra under pointwise multiplication.

($k[\Gamma]$ is of finite type $\Leftrightarrow$ Γ is finite).

in this way one can embed:

$$\begin{array}{ccc} \text{groups} & \hookrightarrow & \text{affine group schemes} \\ \cup & & \cup \\ \text{finite groups} & \hookrightarrow & \text{affine group schemes of finite type} \end{array}$$

Let G be a k-group scheme, and ~~let~~ let A be a k-algebra. Set

$$G(A) := \mathrm{Hom}_{k\text{-alg}}(k[G], A).$$

$\boxed{\text{allow } X \text{ an } k\text{-scheme}}$

Not one geometric object but many geometric objects.

This set is called the A-points of G.

Examples: $\mathbb{G}_a(A) = \mathrm{Hom}_k(k[x], A) = A$.

$\mathbb{G}_m(A) = \mathrm{Hom}(k[x, x^{-1}], A) = A^\times$.

$\mu_n(A) = \mathrm{Hom}(k[x]/(x^n - 1), A) = \{a \in A \mid a^n = 1\}$.

$SL_2(A) = \mathrm{Hom}(-, A) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A^4 \,\middle|\, ad - bc = 1 \right\}$.

Exercise: Check carefully that $G(A)$ is always a group via Δ, S, ε.

This allows us to suddenly to make sense of concepts in group theory, we simply state the group theory definition but require it for all A.

Exercise: If $H \hookrightarrow G$ is a closed embedding then ~~G~~ $H(A) \hookrightarrow G(A)$ is an injective for all A.

Example: ① $N < G$ is a normal subgroup scheme if $N(A) < G(A)$ is normal for all A.

Exercise: Find a group scheme G s.t. $G(k)$ is abelian, but G is not!

② G is ~~commutative~~ abelian if $G(A)$ is ~~commutative~~ abelian for all A.

② Consider the map $Fr: x \mapsto x^p$ on $\mathbb{G}_a$. Is this injective?? in characteristic p.

Note that ~~to~~ for any field k', ~~$(k')^\times \xrightarrow{x \mapsto x^n} (k')^\times$~~

is ~~injective~~, $x^p = (x')^p$

$$\Rightarrow (x - x')^p = 0 \Rightarrow x = x'.$$

~~Hence~~ However, Fr is not injective on arbitrary k-algebras.

In fact, Fr has a kernel α_p defined as follows:

$k[\alpha_p] := k[x]/(x^p)$, $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$.

$S(x) = -x$. Again this is infinitesimal.