Important "points":

$\boxed{\text{FIX INVERSE FROM LAST TIME}}$

X any affine k-scheme, A k-algebra $\quad X(A) = \mathrm{Hom}_{k\text{-alg}}(k[X], A) =$

$$= \mathrm{Hom}_{\text{schemes}}(\mathrm{Spec}\, A, X)$$

the set of A-points of X.

Thus any scheme determines a (covariant) functor k-algebras $\longrightarrow$ sets.

(a) Let $\mathcal{C}$ be any category with products, and let G be a group object in $\mathcal{C}$.

Claim: for any $X \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, G)$ is a group.

f, g

$$X \underset{g}{\overset{f}{\rightrightarrows}} G$$

what is $f * g$

$$\begin{tikzcd} X \arrow[d, "f\times g"] \arrow[dr, "g"] \arrow[dd, bend right, "f"'] & \\ G\times G \arrow[r] \arrow[d] & G \\ G & \end{tikzcd}$$

$$X \xrightarrow{f\times g} G\times G \xrightarrow{m} G.$$

This is "all" we are doing, in the category of affine schemes.

(b) Suppose one would like to show that $f = g$, where f, g are "functions".

In classical mathematics, we would do this by checking $f(x) = g(x)$ for all x.

In classical algebraic geometry $f(x) = \phi(f)$, where $\phi : k[X] \longrightarrow k$ is evaluation at $x \in X$.

But now in the world of affine schemes $f = g$ if and only if $\phi(f) = \phi(g)$ this is true for all homomorphisms $\phi : k[X] \to A$. (tautological, take $A = k[X]$, $\phi = id$!)

This means $f = g$ if this holds at all A-points of X!

Basic topology of group schemes:

A topological space X is irreducible if $X = A \cup B$, A, B closed $\Rightarrow$ one of $A, B = X$.

$$\Updownarrow \qquad\qquad\qquad \Downarrow$$

$$U, V \subset X \text{ open}, \neq \phi \Rightarrow U \cap V \neq \emptyset.$$

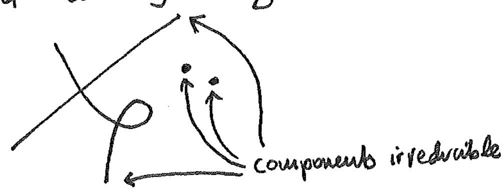
A topological space is Noetherian if any descending sequence of closed subsets $X_1 \supset X_2 \supset \dots$ stabilises.

Lemma: $f : X \to Y$ cts, $A \subset X$ irreducible $\Rightarrow$ $f(A)$ irreducible.

Example: Closed subsets of $\operatorname{Spec} A$ $\Leftrightarrow$ ~~ide~~ $V(I)$ for some ideal $I \subset A$.

Hence $\operatorname{Spec} A$ is Noetherian $\Leftarrow$ A is Noetherian.

Easy Lemma: Any Noetherian space is the union of finitely many maximal irreducible subsets.



The maximal irreducible subsets of X are called {irreducible} components of X.

The irreducible components are the maximal irreducible subsets.

[Proposition: Let G be a group scheme of finite type.

~~G°~~

ⓐ There is a unique irreducible component G° of G the contains the identity.

ⓑ It is a closed normal subgroup of finite index. ⓒ G° is the unique connected component of G containing the identity.

Proof: (i) Suppose X, Y are ~~connected~~ irreducible components of G° containing the identity. Then XY, $\overline{XY}$ are irreducible $\Rightarrow$ $X \subset \overline{XY} \supset Y$, but X, Y are maximal irred $\Rightarrow$ $X = \overline{XY} = Y$.

ⓑ We have seen $XX = X$, also $i(X) = X$, also X is closed. For any other component Y we have $\gamma_c : Y \times X \to G$, $(y, x) \mapsto yxy^{-1}$, then $c(Y \times X)$ is irreducible and contains e hence $\overset{\shortparallel}{X}$. Hence X is normal.

ⓒ

Remark: The map $A \longrightarrow A_{red} = A/\text{nilradical}$ induces $\operatorname{Spec} A_{red} \longrightarrow \operatorname{Spec} A$. This is an ~~isomorphism~~ homeomorphism of topological spaces. Hence when considering topological Q's it is enough to consider reduced schemes.

Corollary: For group schemes of finite type connectedness and irreducibility coincide.

Exercise: The groups $\mathbb{G}_a$, $\mathbb{G}_m$, GL_n, SL_n are connected.

~~Discuss example of μ_n~~

A group scheme is finite if $k(G)$ is finite dimensional over k.

A group scheme is infinitesimal if ~~Spec~~ G has a unique point.

A group scheme G is étale if $k(G)$ is ~~étale~~ an étale k-algebra.

("smooth of relative dimension zero")

Any (finite type) ~~k-algebra~~ étale k-algebra is isomorphic to a ~~finite~~ finite product $k_1 \times \ldots \times k_m$ of finite separable extensions of k. In particular, if $k = \bar{k}$ then an étale k-algebra is a product of copies of $k = \bar{k}$, and hence is of the form $k(\Gamma)$ for some finite (abstract) group Γ.

Example: Consider $\mu_3 = \operatorname{Spec} \mathbb{Q}[X]/(X^3-1)$ over $\mathbb{Q}$.

$$\mathbb{Q}[X]/(X^3-1) \cong \underbrace{\mathbb{Q}[X]/(X-1)}_{\cong\, \mathbb{Q}} \times \underbrace{\mathbb{Q}[X]/(1+X+X^2)}_{\cong\, \mathbb{Q}(e^{2\pi i/3})}.$$

étale $\mathbb{Q}$-algebras

Thm: ~~Any~~ For any group scheme of finite type G there exists a short exact sequence

$$G^0 \hookrightarrow G \twoheadrightarrow H$$

where G^0 denotes the connected component of the identity and H is étale.

Warning: It is subtle to say what a ~~n~~ quotient or exact sequence is. We will avoid this issue.

Example: ~~Show that~~ Assume that k is of characteristic p. Fix $n \geq 1$ and write $n = n' p^k$ with $p \nmid n'$. We have a short exact sequence:

$$\underbrace{\mu_{p^k}}_{\text{infinitesimal}} \hookrightarrow \mu_n \twoheadrightarrow \underbrace{\mu_{n'}}_{\text{étale}}$$

In group theory a fundamental role is played by

- group action,
- representation.

We now want to study these notions in group schemes.

<u>Definition:</u> An affine G-scheme is a scheme X together with a morphism

$$a: \quad G \times X \longrightarrow X$$

such that the diagrams commute:

$$\begin{tikzcd} pt \times X \arrow[r, "\varepsilon \times id_X"] \arrow[dr, bend right, "="'] & G \times X \arrow[d, "a"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} & G \times G \times X \arrow[dl, "id \times a"'] \arrow[dr, "m \times id"] & \\ G \times X \arrow[dr, "a"'] & \circlearrowleft & G \times X \arrow[dl, "a"] \\ & X & \end{tikzcd}.$$

<u>Exercise:</u> ① Show that for any k-algebra A, $X(A)$ is a $G(A)$ set in a natural way.

② Recall SO_2 from last time $k[SO_2] = k[X,Y]/(X^2+Y^2-1)$.

Show that SO_2 acts on $X_\lambda = \{(X,Y) \mid X^2+Y^2=\lambda\}$ for any λ. "by rotation"

Note that $X_\lambda(k)$ might be empty (e.g. $k=\mathbb{R}$, $\lambda<0$).

<u>Q. for audience:</u> Why does G always have a k-point?

... now representations, as this is fundamental we will spend more time on this notion.

Recall that for Γ an (abstract) ~~finite~~ group there is an equivalence between the notion of

Γ-module: ~~and~~ $\Gamma \times M \to M$ s.t. blah

representation: $\rho: \Gamma \to GL(M)$.

3-5

Let M be a k-vector space:

a <u>representation</u> is a homomorphism $\rho : G \to GL(M)$
(of G on M)

(⚡ Need to be a little careful about what this means when $\dim M = \infty$)

a G-module is ... (obvious definition $G \times M \to M$ is not so useful).
by dualising
a $k[G]$-comodule.

A <u>$k[G]$-comodule</u> M is a k-v.s. M together with a map

$$c : M \longrightarrow M \otimes k[G] \qquad \text{"coaction"}$$

such that

$$\begin{tikzcd} M \arrow[r, "c"] \arrow[d, "c"'] & M \otimes k[G] \arrow[d, "id_M \otimes \Delta"] \\ M \otimes k(G) \arrow[r, "c \otimes id_{k[G]}"'] & M \otimes k[G] \otimes k[G] \end{tikzcd}$$

commutes.

<u>Idea:</u> $m \mapsto g \cdot m$ should be a "polynomial in g".

One should think of $\Delta(m) \in M \otimes k[G]$ as a universal recipe for action on M.
$$\text{"} \textstyle\sum m_i \otimes g f_i$$

i.e. for any k-algebra A and $g \in G(A) = \mathrm{Hom}(k[G], A)$

$$g \cdot (m \otimes 1) = \sum m_i \otimes g(f_i) \quad \in \quad M \otimes A.$$

<u>Exercise:</u> G-modules form a reasonable abelian category
(kernels, cokernels, direct sums etc. exist).

<u>Exercise:</u> ① If M is a G-module then $M \otimes A$ is a $G(A)$-module for any A. That is, there exists a homomorphism

$$G(A) \longrightarrow GL_A(M \otimes A) \qquad \text{for any } A.$$

② Show that the category of G-modules and G-representations are equivalent, for finite dimensional M.

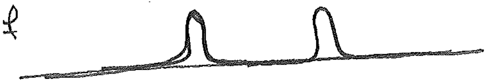
Fundamental principle in representation theory

(1) Suppose $\Gamma \circlearrowright X$ (space, set, ...), $\Gamma \circlearrowright$ (functions, cohomology, ... of X)

non-linear object — linear object.

(2) We can hope to use representation theory to understand X and its Γ-action.

Example: $\mathbb{R} \circlearrowright \mathbb{R}$ by translations. f



For a general function $f : \mathbb{R} \to \mathbb{R}$, $\{ g \cdot f \mid g \in \mathbb{R} \}$ will be an enormous space (e.g. infinite dimensional).

Something miraculous happens for algebraic functions:

$$\lambda \cdot X^n = (X - \lambda)^n \subset \underbrace{\operatorname{span}\{X^0, X^1, \dots, X^n\}}_{D_{\le n}}.$$

In fact, $\mathbb{R} \hookrightarrow D_{\le 0} \subset D_{\le 1} \subset D_{\le 2} \subset \dots$

gives an exhaustive filtration of all polynomial functions via finite-dimensional representations.

Exercise: Write down the corresponding representations $(\mathbb{R}, +) \to GL_n(\mathbb{R})$.

Example 2: $\mathbb{C}^\times \circlearrowright \mathbb{C}^\times$ via multiplication.

A general function will generate a complicated infinite dimensional rep of $\mathbb{C}^\times$.

But for any $m \in \mathbb{Z}$: $\lambda \cdot X^m = (\lambda X)^m = \lambda^m X^m$.

Hence $\mathbb{C}[\mathbb{C}^\times] = \mathbb{C}[X, X^{-1}] = \bigoplus_{m \in \mathbb{Z}} \mathbb{C} X^m$

is a direct sum of irreducible representations!

Exercise: These are all the irred. reps of $\mathbb{G}_m$.

This ~~is a general~~ "finite-dimensionality" is a general feature of representations of group schemes.

Finiteness

Lemma: Let M be a G-module. Then any finite-dimensional subspace $V \subset M$ is contained in a finite-dimensional G-submodule.

~~Moreover,~~

Proof: It is enough to prove this when $V = \langle m \rangle$ is one-dimensional.

~~Let $c(m) = \sum m_i \otimes f_i$. We claim~~

with the ~~f_i some~~

~~Choose a basis $f_1, f_2, \ldots$ for $k[G]$ and write $c(m) = \sum m_i \otimes f_i$.~~

~~We claim $\langle m_i$~~

Write $c(m) = \sum_{i=1}^{k} m_i \otimes f_i$ for linearly independent f_i.

Claim: $M = \langle m_i \rangle_{i=1}^{k}$ is a G-submodule.

Extend $f_1, \ldots, f_k$ to a basis $f_1, \ldots, f_k, f_{k+1}, \ldots$ of $k[G]$.

Write $\Delta(f_i) = \sum f_{ik} \otimes f_k$ for certain $f_{ik} \in k[G]$.

Then
$$(c \otimes id)(c(m)) = (id \otimes \Delta)(c(m))$$
$$\sum c(m_i) \otimes f_i = \sum_{i,k} m_i \otimes f_{ik} \otimes f_k$$

and hence $c(m_k) = \sum_{i=1}^{k} m_i \otimes f_{ik}$ which proves the proposition. □

$$m = (id \otimes \varepsilon)(c(m)) = \sum_{i=1}^{k} m_i \, \varepsilon(f_i) \Rightarrow m \in \langle m_i \rangle.$$

The most fundamental G-module is the <u>right-regular representation</u>:

$$M := k[G] \qquad c = \Delta : M \longrightarrow M \otimes k[G].$$

<u>Rmk</u>: This is an algebraic group version of $\mathrm{Fun}(G)$, $g \cdot f(x) = f(xg)$.

<u>Lemma</u>: The right reg rep contains a faithful finite-dimensional representation.

(<u>faithful</u>: the corresponding morphism $G \to GL(M)$ is injective on A-points for all A.)

<u>Proof</u>: By finiteness lemma there exists $M \subset k[G]$ s.t.

① M is finite dim.

② M generates $k[G]$ as an algebra.

We now show that M is faithful. Let A be a k-algebra and $g \in G(A)$ s.t.

$$g \cdot (m \otimes 1) = (\mathrm{id} \otimes g)(c(m)) = m \otimes 1 \quad \forall m \in M.$$

$\Longrightarrow$ i.e.

$$\begin{tikzcd}
M \otimes A \arrow[r, "c \otimes \mathrm{id}_A"] \arrow[d, phantom, "\cap"] \arrow[rr, bend left, "\mathrm{id}"] & M \otimes k[G] \otimes A \arrow[r, "\mathrm{id} \otimes g"] \arrow[d, phantom, "\cap"] & M \otimes A \arrow[d, phantom, "\cap"] \\
k[G] \otimes A \arrow[r, "c \otimes \mathrm{id}_M"] \arrow[rr, bend right, "\mathrm{id}"'] & k[G] \otimes k[G] \otimes A \arrow[r, "\mathrm{id} \otimes g"] & k[G] \otimes A
\end{tikzcd}$$

$\left(\begin{array}{l} M \otimes A \text{ generates} \\ k[G] \otimes A \end{array} \right)$.

<u>Exercise</u>: $\Longrightarrow$ $g = \varepsilon \otimes \mathrm{id}$.

$\square$