Website: www.maths.usyd.edu.au/u/geordie/GroupSchemesAndRepresentations/

Recall from last time: G groupe scheme (= affine k-group scheme).

- a representation is a homomorphism $\rho: G \to GL(V)$ for some k-vector space V (care needed when V is ∞-dim ...)

- a G-module is a $k(G)$-comodule, i.e. $c: M \to M \otimes k(G)$ s.t.

$$\begin{tikzcd} M \arrow[r, "c"] \arrow[d, "c"'] & M\otimes k(G) \arrow[d, "id\otimes\Delta"] \\ M\otimes k(G) \arrow[r, "c\otimes id"] & M\otimes k(G)\otimes k(G) \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "c"] \arrow[dr, "\sim"'] & M\otimes k(G) \arrow[d, "id\otimes\varepsilon"] \\ & M\otimes k \end{tikzcd}$$

Passage from one to the other: comodule is a "universal formula for g-action"

Choose a basis e_i for M and write $c(e_j) = \sum e_i \otimes a_{ij}$, with $a_{ij} \in k(G)$.

Then $g \mapsto \begin{pmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \\ & & \ddots \end{pmatrix}$ defines a representation of G.

<u>Exercise:</u> Check carefully that this is true. That is that $x_{ij} \mapsto a_{ij}$ defines a morphism of Hopf algebras

$$k(GL_n) \to k(G).$$

Let Γ denote a finite group. V a v.s. over k.

<u>Equivalent notions:</u> • $\Gamma \to GL(V)$, • Γ-module V.

If $k = \mathbb{C}$, any V is semi-simple, $V = \bigoplus V_i$, V_i irreducible.

Any V_i is determined by its character: $g \mapsto \mathrm{Tr}_{V_i}(\rho(g))$

⋮

~~For any (large enough) field of char p, there is a reduction modulo~~

In rough terms, the same is true over any field of char p, with $p \nmid |\Gamma|$.

Recall isomorphism: $k((\mathbb{Z}/p\mathbb{Z})^n) \cong k[X_1, \dots, X_n]/(X^p)$.

If $p \mid |\Gamma|$ story is much more complicated.

Eg. Rep of $\mathbb{Z}/p\mathbb{Z}$ in char p $\simeq$ reps of $k[x]/(x^p)$ "understood".

However, rep. theory of $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ is quite complicated.

(Eg. $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ has indecomposable modules of arbitrary dimension in char. 2)

Two very important Remarks: Suppose G of finite type.

ⓐ ~~If k = k̄ ... then~~ Any representation

$$\rho : G \longrightarrow GL(M)$$

gives rise to a homomorphism $G(k) \xrightarrow{\rho(k)} GL(M)$. If $k = \bar{k}$ then two representations ρ_1, ρ_2 are isomorphic if and only if $\rho_1(k)$ and $\rho_2(k)$ are. ~~Thus we obtain~~ In fact we obtain a fully faithful embedding:

$$\underset{\substack{\text{(reps of group}\\ \text{scheme } G)}}{\operatorname{Rep} G} \hookrightarrow \underset{\substack{\text{(reps of abstract group } G(k)\\ \text{on } k\text{-vector spaces)}}}{\operatorname{Rep}_k G(k)}.$$

Roughly speaking, the image consists of those representations which are "given by formulas".

This is very far from being an equivalence in general.

E.g. $(\overline{\mathbb{Q}})^\times \cong \prod_{n \geq 2} \mathbb{Z}/n\mathbb{Z} \times \prod_{i \in I} \mathbb{Z}$. (I guess this is correct?)

ⓑ Suppose k is a finite field ($k = \mathbb{F}_p$) and G is of finite type. Then $\Gamma_q = G(\mathbb{F}_q)$ (any power q of p) is a finite group. Hence we have a functor

$$\operatorname{Rep} G \longrightarrow \operatorname{Rep}_k \Gamma_m .$$

This will ~~not be~~ never be fully faithful, but is nonetheless an extremely interesting functor!

"Compatible family of representations".

(1) Comparison w/ finite group: Work over $\mathbb{F}_p$.

$$\mathbb{G}_m(\mathbb{F}_q) = \mathbb{F}_q^\times \cong \quad = \langle \gamma \rangle \xleftarrow{\sim} \mathbb{Z}/(q-1)$$
$$\gamma \longleftarrow 1$$

Representations of $\mathbb{Z}/(q-1)\mathbb{Z}$ in char p is semi-simple $(p \nmid (q-1))$.

(2) $\mathbb{G}_a(\mathbb{F}_q) \cong (\mathbb{Z}/p\mathbb{Z})^m$.

Representations of $(\mathbb{Z}/p\mathbb{Z})^m \xrightarrow{\sim}$ Reps of $k[X_1, \dots, X_m]/(X_i^p)$.

Hence rep theory of $\mathbb{G}_a$ and $\mathbb{G}_a(\mathbb{F}_q)$ are quite similar.

(Frobenius twisting corresponds to shifting indices in $\mathbb{G}_a$, but corresponds to "cycling" in $\mathbb{G}_a(\mathbb{F}_q)$, ... hence ? correspondence can't be made too precise.)

<u>Theorem</u>: Any group scheme of finite type admits a closed embedding inside some GL_n.

<u>Before remarks</u>: Given a comodule $V \to V \otimes k[G]$ we have seen that the corresponding representation is given by $(c_{ij})_{i,j=1}^n$.

<u>Recall</u>: A morphism $H \to G$ of algebraic groups is a <u>closed embedding</u> if $k[G] \to k[H]$ is a surjection of Hopf algebras.

Rmk: $\Longrightarrow H(A) \to G(A)$ is injective for all k-algebras A. In fact, this is if and only if, but this is a theorem.

Representations of $\mathbb{G}_m$: (was an exercise)

$$c: V \longrightarrow V \otimes k[x^{\pm 1}]$$

$$c(v) = \sum s_i(v) \otimes x^i.$$

$(\mathrm{id} \otimes \varepsilon) \circ c = \mathrm{id} \implies \sum_i s_i(v) = v$ (i.e. $\sum_i s_i = \mathrm{id}$).

$(c \otimes \mathrm{id}) \circ c = (\mathrm{id} \otimes \Delta) \circ c \implies$ ...

$$\sum s_k(s_i(v)) \otimes x^k \otimes x^i = \sum_i s_i(v) \otimes x^i \otimes x^i.$$

Hence: $s_k s_i = 0$ unless $i = k$ in which case $s_k^2 = s_k$.

In other words, giving $\mathbb{G}_m$-action is the same as giving $V = \bigoplus V_i$ with $p_i =$ projection to V_i.

Rmk: This category is semi-simple always!

Representations of μ_n: Same calculations show that a rep of μ_n is the same thing as a $\mathbb{Z}/n\mathbb{Z}$-grading on V.

(This is also true in char p, when μ_n is not reduced)

Rmk: The morphism $\mu_n \to \mathbb{G}_m$ defines a restriction map.

This gives the "wrap up the grading" functor.

AFTER $\mathbb{G}_a$ DISCUSSION.

Exercise: In char p, $\mathbb{G}_a$ has non-reduced closed group subschemes α_{p^k}, $k[X]/(X^{p^k})$. Show that a representation of α_{p^m} is the same thing as m-commuting nilpotent operators of nilpotently degree $= p$.

Representations of G_a: Let V be a G_a-module.

That is, V is given by $c: V \to V \otimes k[X]$.

Write $c(v) = \sum_i \delta_i(v) \otimes X^i$.

We now expand what the conditions to be a co-action are:

$$(c \otimes id) \circ c = (id \circ \Delta) \circ c \Rightarrow \sum \delta_k(\delta_i(v)) \otimes X^k \otimes X^i$$

$$\shortparallel$$

$$\Rightarrow \delta_a(\delta_b(v)) = \binom{a+b}{a} \delta_{a+b}. \qquad \sum_i \delta_i(v) \otimes (X \otimes 1 + 1 \otimes X)^i$$

$$\shortparallel$$

$$\sum_j \binom{i}{j} \delta_i(v) \otimes X^{i-j} \otimes X^j.$$

$(id \circ \varepsilon) \otimes c = id \Rightarrow \delta_0 = id.$

Hence we see that a representation of G_a on V is the same as:

- operators $\delta_1, \delta_2, \ldots$ $\delta_i : V \to V$ s.t. finitely many are $\neq 0$.
- such that $\delta_i \delta_j = \binom{i+j}{i} \delta_{i+j}$.

OK

Two cases: characteristic of k is zero. Then $n! \delta_n = \delta_1^n$

$\delta_{i+j} = \cancel{\frac{1}{(i+j)}}$ and hence $\delta_n = \frac{1}{n!} \delta_1$

and a representation of G_a is the same thing as a nilpotent endomorphism δ_1.

$$\delta_1 \delta_p = \overset{\neq 0}{(p+1)} \delta_{p+1} = \delta_p \delta_1.$$

Char $k = p > 0$: In this case $\overset{0}{=} p! \circ \delta_p = \delta_1^p$,

and hence δ_p is not recoverable from δ_1. In this case write $A_i := \delta_{p^i}$ ~~and for~~, each A_i is satisfies $A_i^p = p$

and if $n = \sum d_i p^i$ (p-adic expansion), then with $0 \leq d_i < p$

$$\delta_n = \prod \left(\frac{1}{d_i!}\right) A_i.$$

In particular, if c_{ij} generate $k[G]$ then the corresponding map $G \to GL_n$ is a closed embedding.

Recall: <u>Finiteness lemma:</u> Let M be a $k[G]$-comodule. Any f.d. $V \subset M$ is contained in a finite dimensional $k[G]$-comodule, V'.

Now consider "right regular representation" $M = k[G]$ w/

$$c = \Delta : M \to M \otimes k[G].$$

Let $V \subset k[G]$ be any f.d. subspace which generates $k[G]$ as a k-algebra. ~~Then, 2 if~~ let V' denote a f.d. sub-$k[G]$-comodule.

Then $G \hookrightarrow GL(V')$ is a closed embedding.

Fun(G)
some space of functions on G
(e.g. continuous, all, algebraic).
$G \times G$-module via...

<u>Matrix coefficients:</u> Let G be a group (finite, Lie, algebraic). and let $\rho : G \to GL(V)$ be a representation.

Recall: V^* is a representation via $(g \cdot w)(v) = w(g^{-1}v)$.

Let $v \in V$, $w \in V^*$ and consider $\phi_{v,w}(g) := w(gv)$ a function on G. Called a <u>matrix coefficient</u>.

Now: $$(a \cdot \phi_{w,v} \cdot b)(g) = \phi_{v,w}(bga) = w(bga \cdot v)$$
$$= (b^{-1} \cdot w)(g \cdot (a \cdot v))$$
$$= \phi_{b^{-1}w,\, av}(g).$$

<u>Rmk:</u> Why is this called a matrix coefficient?

In other words: $\{ \phi_{w,v} \mid w \in V^*, v \in W \} \subset \mathrm{Fun}(G)$

defines a subspace stable under the left and right action of G on itself.

Also, for w fixed the map $v \mapsto \phi_{w,v}$ defines a morphism of representations $V \to \mathrm{Fun}(G)$.

We now do an algebraic groups version. ~~Fix [illegible]~~*

Let M be a G-module, and fix $w \in M^*$.

We define $\phi_w : M \longrightarrow k[G]$ via $M \xrightarrow{c} M \otimes k[G] \xrightarrow{w \otimes \mathrm{id}} k[G]$.

<u>Lemma:</u> ϕ_w is a map of G-modules. (i.e. of $k[G]$-comodules)

<u>Proof</u>:

$$
\begin{tikzcd}
M \arrow[r, "c"] \arrow[d, "\phi_w"'] \arrow[dr, "\phi_w"] & M \otimes k[G] \arrow[r, "{c \otimes \mathrm{id} \,=\, \mathrm{id} \otimes \Delta}"] \arrow[d, "\phi_w \otimes \mathrm{id}"] \arrow[dr] & M \otimes k[G] \otimes k[G] \arrow[d, "\phi_w \otimes \mathrm{id} \otimes \mathrm{id}"] \\
k \arrow[r] & k[G] \arrow[r, "\Delta"] & k[G] \otimes k[G]
\end{tikzcd}
$$

$\square$

<u>Corollary</u>: Suppose that $\{w_i\}_{i \in I}$ generate M^*. Then the map $M \xrightarrow{\oplus \phi_{w_i}} \bigoplus_{i \in I} k[G]$ ~~ge~~ embeds M inside the regular rep.

<u>Rmk</u>: One may also check rather easily that $k[G]$ is an injective G-module. Thus corollary shows that $k[G]$ is an injective generator of Rep G.