What group schemes have we met so far?

(1) $\mathbb{G}_m = \operatorname{Spec} k[x^{\pm 1}]$ w/ closed subschemes $\mu_n = \operatorname{Spec} k[x]/(x^n - 1)$ for all $n \geq 1$.

(2) $\mathbb{G}_a = \operatorname{Spec} k[x]$

$\xrightarrow{\text{char } 0}$ **Exercise:** no non-trivial closed subschemes

$\xrightarrow{\text{char } k = p}$ $\alpha_{p^m} = \operatorname{Spec} k[x]/(x^{p^m})$ $\forall m \geq 1$.

(3) The "circle" $\{x^2 + y^2 = 1\} \subset k^2$.

(4) SL_n, GL_n.

(5) Any group $\Gamma \rightsquigarrow$ group scheme $k[\Gamma]$ finite type if Γ is finite.

Last time: (1) $\operatorname{Rep} \mathbb{G}_m \xrightarrow{\sim}$ $\mathbb{Z}$-graded vector spaces.

Exercise: (a) $\operatorname{Rep} \mu_n \xrightarrow{\sim}$ $\mathbb{Z}/n\mathbb{Z}$-graded vector spaces

(b) Describe $\operatorname{Rep} \mathbb{G}_m \to \operatorname{Rep} \mu_n$ ("restriction").

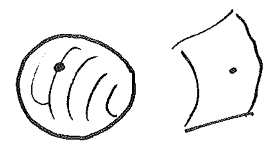
(2) $\operatorname{Rep} \mathbb{G}_a \xrightarrow{\sim}$
$\begin{cases} \text{pairs } (V, \phi),\ V \text{ v.s., } \phi \in \operatorname{End}(V) \text{ s.t. } \phi \text{ is nilpotent} & \text{char } k = 0. \\ \text{tuples } (V, \phi_1, \phi_2, \dots),\ V \text{ v.s.} \\ \quad \phi_i \in \operatorname{End}(V),\ \phi_i \phi_j = \phi_j \phi_i\ \forall i, j. & \text{char } k = p. \\ \quad \phi_i^p = 0 \quad \text{(very complicated)} & \end{cases}$

Exercise: (a) $\operatorname{Rep} \alpha_{p^n} = \{$ tuples $(V, \phi_1, \dots, \phi_n)$ satisfying conditions above.

(b) Show that $\operatorname{Rep} \mathbb{G}_a \to \operatorname{Rep} \alpha_{p^n}$ is given by "forget $\phi_{n+1}, \phi_{n+2}, \dots$".

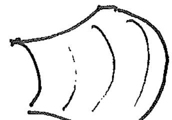
In order to get further (even to analyse Rep SL_2) we need more tools.

Infinitesimal Theory: Tangent space, ~~[illegible]~~ Lie algebra, distributions.



Let M be a manifold and $x \in M$ be a point.

(e.g. $M \subset \mathbb{R}^n$ smooth).



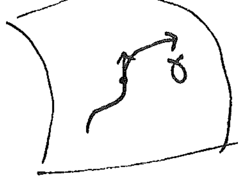
$T_x M$

Two ways to understand tangent space of M at x:

ⓐ "velocity of curves": equivalence classes of maps "up to order 1"

$$\gamma : (-\varepsilon, \varepsilon) \longrightarrow M \quad \text{s.t.} \quad \gamma(0) = x.$$

$(-\varepsilon,\varepsilon) \subset \mathbb{R}$



ⓑ as "derivations at x": $D: \ C^\infty(M) \longrightarrow \mathbb{R}_x$

$$(\text{i.e.} \quad D(fg) = f(x)D(g) + D(f)g(x).)$$

$\mathbb{R}$-linear maps

Passage from ⓐ → ⓑ: take the derivative along γ.

$\boxed{\text{Def: } k[\varepsilon] := k[\varepsilon]/(\varepsilon^2).}$

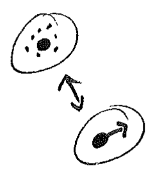
Remark: I will use ε for count and dual numbers.

In algebraic geometry: (X an affine scheme / $k = \bar{k}$, $x \in X(k)$).

" $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ s.t. $\gamma(0) = x$ "

$$\begin{tikzcd} \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \arrow[r, "\gamma"] & X \\ \operatorname{Spec} k \arrow[u] \arrow[ur, bend right, "x"'] & \end{tikzcd}$$

Think of $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ "as being a point with an infinitesimal direction".

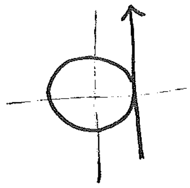


We can rephrase via functor of points: $k[\varepsilon] \twoheadrightarrow k$ induces

$$\begin{tikzcd} \gamma \in X(k[\varepsilon]) \arrow[d, "p"] \\ x \in X(k) \end{tikzcd} \qquad T_x X = \{\gamma \mid p(\gamma) = x\}.$$

<u>Example:</u> $\text{Circle} = \{X^2+Y^2=1\} \subset k^2$.

$$T_{(1,0)}\,\text{Circle} = \{(1+\varepsilon u, 0+\varepsilon v) \mid (1+\varepsilon u)^2 + (\varepsilon v)^2 = 1 \Leftrightarrow u = 0.\}$$



$$= \{(1, \varepsilon v)\}.$$

<u>Rephrase in algebra:</u>

$$\begin{tikzcd} k[\varepsilon] & & k[X] \arrow[ll, "\gamma"'] \arrow[dl, "x"] \\ & k \arrow[from=ul, "\varepsilon=0"'] & \end{tikzcd}$$

Of course we can write

$$\gamma(f) = x(f) + \gamma_1(f)\varepsilon.$$

γ is a homomorphism $\Longleftrightarrow$ γ_1 is k-linear and

$$\gamma(fg) = \gamma(f)\gamma(g)$$

$$\gamma(fg) = (x(fg) + \gamma_1(fg)\varepsilon)$$

$$\gamma(f)\gamma(g) = (x(f) + \gamma_1(f)\varepsilon)(x(g) + \gamma_1(g)\varepsilon) = x(f)x(g) + (x(g)\gamma_1(f) + x(f)\gamma_1(g))\varepsilon$$

$$\Updownarrow$$

γ_1 is a k-derivation.

<u>Hence:</u>

<u>Lemma:</u> $T_x X \simeq \mathrm{Der}_k(k[X], k_x) \cong (I_x / I_x^2)^*$

k_x: k viewed as a $k[X]$-module via x.

<u>Exercise:</u> Let $I_x = \ker(k[X], k)$. Show that $T_x X \cong (I_x/I_x^2)^*$.

I_x / I_x^2 is often called <u>Zariski tangent space</u>. c.f. Zariski "On the notion ..."

$pt = \operatorname{Spec} k$

Exercises:

① $T_{pt}\, pt = \{0\}$.

② $T_0 \operatorname{Spec} k[\varepsilon] = k$.

③ $X = (y^2 = x^2(x+1))$

Compute $T_{(0,0)} X$, $T_{(0,1)} X$.

④ Let $K = \operatorname{Spec} k(x)$ and $L = \operatorname{Spec} k(x^{1/p})$, $\operatorname{char} k = p$.

Consider $X = \operatorname{Spec} L$ as a scheme over $\operatorname{Spec} K$.

Show that $T_0 X = K$. "smoothness is relative".

Example: Write $e = \mathrm{id} \in SL_n$.

ⓐ $T_e SL_n = \{ (\mathrm{id} + \varepsilon X) \mid \det(\mathrm{id} + \varepsilon X) = 1 \}$

$$= \{ 1 + (X_{11} + X_{12} + \dots + X_{nn}) = 1 \} = \{ X \in Mat_{n\times n} \mid \operatorname{Tr} X = 0 \}.$$

ⓑ $T_e GL_n = \{ (\mathrm{id} + \varepsilon X) \mid \det(\mathrm{id} + \varepsilon X)$ is invertible in $k[\varepsilon][X_{11}, \dots, X_{nn}] \}$.

$$= \{ X \in Mat_{n\times n} \}.$$

The Lie Algebra of a Lie Group (heuristic)

Suppose G is a Lie group. (E.g. $G = SO_2 = S^1 =$ rotations of $\mathbb{R}^2$,

$= \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$

$SO_3 =$ rotations of $\mathbb{R}^3$ and reflections

$= \{ A \in GL_3 \mid AA^t = 1 ,\ \det A = 1 \}$

$SL_2(\mathbb{R})$).

$\boxed{\text{Normally (without extra structure) we cannot map } T_x M \to M.}$

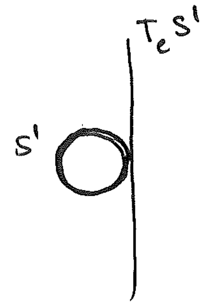
But for a Lie group we can ~~extend~~ represent any $\gamma : (-\varepsilon, \varepsilon) \to G$ with $\gamma(0) = G$

by a <u>homomorphism</u> $\tilde{\gamma} : \mathbb{R} \to G$ "one parameter subgroup".

This gives us a map: $\exp : T_e G \to G$ "exponential map".
(evaluation at $\gamma(1)$) which is a local homeomorphism in a nbhd of the identity.

E.g: If $G = S^1 = \{ z \in \mathbb{C} \mid \|z\| = 1 \}$ then $T_e S^1 = i\mathbb{R}$ and

$\exp(\gamma) = e^{\gamma}$. exp



$T_e G$ has the structure of a <u>Lie algebra</u>.

A Lie algebra over a field k is a k-v.s. $\mathfrak{g}$ with a bilinear bracket:

$[-,-] : \mathfrak{g} \times \mathfrak{g} \to k$

s.t. ① $[x, x] = 0 \quad \forall x \in \mathfrak{g}$

② $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \quad \forall x, y, z \in \mathfrak{g}$.

("Jacobi identity").

Remark: (a) $0 = [x+y, x+y] \Rightarrow [x, y] = -[y, x] \quad \forall x, y \in \mathfrak{g}$.

A morphism of Lie algebras is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ s.t.

$$\phi([x,y]) = [\phi(x), \phi(y)], \quad \forall x, y \in \mathfrak{g}.$$

Example: If A is an associative algebra over k (~~possibly non-associative~~), then $[a,b] := a \cdot b - b \cdot a$ defines a Lie algebra structure on A.

Two ways of understanding the Lie bracket on Lie $G = T_eG$:

① $G \circlearrowright G$ conjugation, fixing $e \in G$.

Hence $G \circlearrowright \underbrace{T_eG}_{\mathfrak{g}}$. $\rightsquigarrow$ differentiate: $\underbrace{\text{Lie } G}_{\mathfrak{g}} \circlearrowright \underbrace{\text{Lie } G}_{\mathfrak{g}}$

yields bracket: $x \mapsto [x, -]$.

② One can identify Lie G = left invariant vector fields on G.

$\cap$

vector fields on G.

(Lie bracket of vector fields gives Lie bracket on Lie G).

③ In a nbhd of the ~~identity~~ $0 \in$ Lie G one has.

$$\exp(x)\exp(y) = \exp(x + y + \tfrac{1}{2}[x,y] + \dots)$$

↖ all terms involve Lie bracket.

"Baker-Campbell-Hausdorff formula".

In other words, the Lie bracket completely describes the product in G, if G is connected.

The Lie algebra of $GL(V)$ is $\underbrace{\mathfrak{gl}(V)}_{= \mathrm{End}(V)}$ with Lie bracket $[A,B] = AB - BA$.

Main Theorems:

If G is connected,
Differentiation provides a fully-faithful functor

$$\mathrm{Rep}_{f.d.}\, G \longrightarrow \mathrm{Rep}\, \mathfrak{g}.$$

Let $A = k(G)$ and consider the space $\mathrm{Der}_k(A,A)$ "vector fields on ~~…~~ G".

<u>Check:</u> If $D, D' \in \mathrm{Der}_k(A,A)$, then $[D,D'] = D\circ D' - D'\circ D$ is again a derivation. Hence $\mathrm{Der}_k(A,A)$ is a Lie algebra.

<u>Def:</u> A derivation is <u>left-invariant</u> if

$$\begin{tikzcd} A \arrow[r, "D"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, "\Delta"] \\ A\otimes A \arrow[r, "id\otimes D"] & A\otimes A \end{tikzcd}$$

commutes.

<u>Check:</u>

$$\Delta\circ([D,D']) = \Delta\circ D\circ D' - \Delta\circ D'\circ \cancel{\otimes} D = (id\otimes(D\circ D'))\circ\Delta - (id\otimes D'\circ D)\circ\Delta$$
$$= id\otimes [D,D'].$$

Hence left-invariant derivations form a Lie subalgebra $\subset \mathrm{Der}_k(A,A)$.

<u>Proposition:</u> $D \mapsto \varepsilon\otimes D : \mathrm{Der}_k(A,A) \longrightarrow \mathrm{Der}_k(A,k) = T_eG$

~~deltin~~ induces an isomorphism from the space of left invariant derivations onto $\mathrm{Der}_k(A,k)$.

<u>Proof:</u>

$$\underset{\substack{\uparrow \\ \text{"counit"}}}{D =} (id\otimes\varepsilon)\circ\Delta\circ D \underset{\substack{\uparrow \\ \text{"left-invariance"}}}{=} (id\otimes\varepsilon)\circ(id\otimes D)\circ\Delta = (id\otimes(\varepsilon\circ D))\circ\Delta.$$

Hence D is determined by ~~d~~ $\varepsilon\circ D$.

Conversely, if d is a derivation, then $D = (id\otimes d)\circ\Delta$ defines a left-invariant derivation $A\to A$. $\square$