A X k-scheme (6-1)
$x \in X(A)$ A-point:
$T_x X := \{ \overset{\gamma}{p} \in X(A[\varepsilon])$ s.t. $\gamma \circ p = x \in X(A) \}$.
$\parallel$
$\mathrm{Der}(k[X], A)$.

Last time: M manifold, $x \in M$ pt.

tangent space of M at x $\to$ velocity of curves $\gamma : (-\varepsilon, \varepsilon) \to M$ s.t. $\gamma(0) = x$.

$\to$ $\mathbb{R}$-derivations $\mathrm{Der}(C^\infty(M), \mathbb{R}_x)$

$\boxed{\text{Philosophical comment about analogy in mathematics!}}$

G group scheme (affine k group scheme):

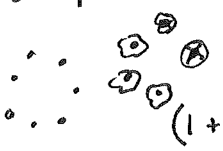
$$\mathfrak{g} = \mathrm{Lie}\, G = T_{id} G = \ker\left(G(k[\varepsilon]) \to G(k) \right).$$

Eg: $\mathrm{Lie}\, \mu_n = \begin{cases} \{0\} & \text{if } p \nmid n, \\ k & \text{if } p \mid n \end{cases}$ if char p

$\mathrm{Lie}\, \mathbb{G}_a = \ker\left(k[\varepsilon] \xrightarrow{\varepsilon = 0} k \right) = k\varepsilon \simeq k$.

$\mathrm{Lie}\, SL_n = \{ X \in \mathrm{Mat}_{n \times n}(k) \mid \mathrm{Trace}(X) = 0 \}$.

$\mathrm{Lie}\, \mathbb{G}_m = \{ (1 + \gamma\varepsilon) \in k[\varepsilon]^\times \} \cong k$.



$(1 + \varepsilon\gamma)^n = 1$, $1 + n\varepsilon\gamma = 0 \Rightarrow \gamma = 0$ if $p \nmid n$.

$p \nmid n$

<u>Exercise:</u> ① Show that $T_{id} G$ is a k-vector space in a natural way.

Show that the ~~k-algebra structure~~ addition on $T_{id} G$ agrees with the group structure via $T_{id} G = \ker\left(G(k[\varepsilon]) \to G(k) \right)$.

$\boxed{\text{Rmk: How to think about a finitely presented group, parallel to Lie groups!}}$

If G is a Lie group (group object in manifolds) then $\mathrm{Lie}\, G$ carries a Lie bracket.

"infinitesimal shadow of multiplication" $[-,-] : \mathrm{Lie}\, G \times \mathrm{Lie}\, G \to \mathrm{Lie}\, G$ making $\mathrm{Lie}\, G$ a Lie algebra.

<u>Remark:</u> For Lie there was no difference between a Lie group and its Lie algebra of "infinitesimal transformations". We owe the term "Lie algebra" to Weyl (1934), who was the first to understand global Lie groups as manifolds.

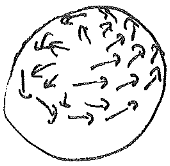
The Lie algebra is an essential tool in studying representations of G.

<u>Goal:</u> construct the Lie ~~algebra~~ bracket on $\mathrm{Lie}\, G$ for G a group scheme.

Remark: Weil, "On analogy in Mathematics", 1941.
(letter to sister Simone Weil from prison).

M a manifold, vector field on M $\nearrow$ section of tangent bundle (geometric) (i.e. assign a tangent vector to each point $x \in M$)

$\searrow$ $\mathbb{R}$-derivation $D : C^\infty(M) \to C^\infty(M)$ (algebraic)



Vector fields form a $C^\infty(M)$-module.

$\boxed{\text{Recall definition of derivation.}}$

Let X be an affine scheme. A vector field D on X is a k-derivation

$$D : k[X] \to k[X].$$

Recall that the tangent space of X at ~~as~~ an A-point x is $\mathrm{Der}(k[X], A)$.

Hence any vector field yields a point in $T_x X$ for any A-point x.

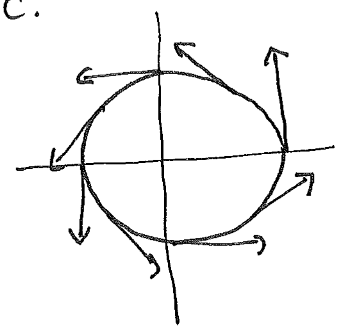
Example: (a) Let $\mathbb{A}^2 = \mathrm{Spec}\ \overset{A}{\overset{\shortparallel}{k[X,Y]}}$. Any derivation is determined by its values on X, Y and hence $\mathrm{Vect}(\mathbb{A}^2) = \cancel{A\frac{\partial}{\partial X} \oplus A\frac{\partial}{\partial Y}}$

$$= k[X,Y]\frac{\partial}{\partial X} \oplus k[X,Y]\frac{\partial}{\partial Y}$$

(b) Let $C = \mathrm{Spec}\ k[X,Y]/(X^2+Y^2-1)$ "circle".

Then $\mathrm{Vect}(C) = \left(k[X,Y]\frac{\partial}{\partial X} \oplus k[X,Y]\frac{\partial}{\partial Y}\right)\left(2\left(X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y}\right)\right)$.

Hence, e.g. $Y\frac{\partial}{\partial X} - X\frac{\partial}{\partial Y}$ is a vector field on C.



In fact, $\mathrm{Vect}(C)$ is free of rank 1 on this generator.

(c) Exercise: What are the vector fields on $\mathrm{Spec}\ k[X]/(X^n)$?

(should see two distinct cases ...)

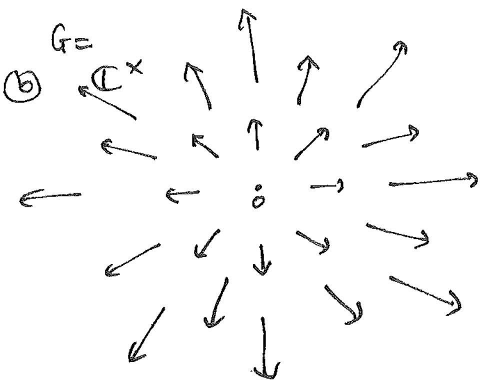
Suppose that G is a ~~group~~ Lie group ~~(abstract)~~ Lie}. The left-translation operator (for $g \in G$) is 6-3

$$T_g(f)(x) = f(gx).$$

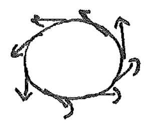
A vector field is *left-invariant* if (when viewed as a derivation) it commutes with T_g for all $g \in G$.

Eg: (a) $G = (\mathbb{R}, +)$

(b) $G = \mathbb{C}^\times$



(c) $G = S^1$:



Now rephrase in terms of Hopf algebras:

given $g \in G(k) = \mathrm{Hom}(k[G], k)$, left translation is the operator:

$$\begin{tikzcd} k[G] \arrow[r, "\Delta"] \arrow[d, "D"] & k[G] \otimes k[G] \arrow[r, "g \otimes id"] \arrow[d, phantom, "\circlearrowright"] & k \otimes k[G] = k[G] \arrow[d, "D"] \\ k[G] \arrow[r] & k[G] \otimes k[G] \arrow[r] & k \otimes k[G] \cong k[G] \end{tikzcd}$$

Hence an operator $D: k[G] \to k[G]$ commutes with T_g if this diagram commutes.

If it commutes for all A-points then

$$(*) \qquad \begin{tikzcd} k[G] \arrow[r, "\Delta"] \arrow[d, "D"] \arrow[dr, phantom, "\circlearrowright"] & k[G] \otimes k[G] \arrow[d, "id \otimes D"] \\ k[G] \arrow[r] & k[G] \otimes k[G] \end{tikzcd}$$

commutes.

Definition: An operator $D: k[G] \to k[G]$ is *left-invariant* if $(*)$ commutes.

Eg: Suppose $D = f \frac{\partial}{\partial X}$ on $k[G_m] = k[X^{\pm 1}]$ is left-invariant.

Then

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, mapsto] & X \otimes X \arrow[d, mapsto] \\ f \arrow[r, mapsto] & \Delta(f) \quad X \otimes f \end{tikzcd}$$

Hence $f = X$.

Similarly:

$\mathrm{LeftInvariant}(G_a) = k \frac{\partial}{\partial X}$

" $(C) = k\left(Y \frac{\partial}{\partial X} - X \frac{\partial}{\partial Y}\right)$.

Lemma: We have isomorphisms:

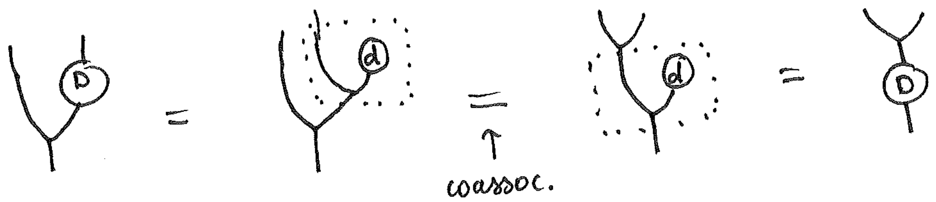
$$\begin{tikzcd}
\boxed{\mathrm{End}(k[G])} & \\
\cup & \\
\lambda : \mathrm{Left}(k[G]) \arrow[r, "f \mapsto (\varepsilon\circ f)", "\sim"'] \arrow[d, phantom, "\cup"] & \mathrm{Hom}(k[G],k) \arrow[d, phantom, "\cup"] \\
\text{left invariant vector fields on } G \arrow[r, "\sim"] & \mathrm{Der}(k[G],k) = \mathrm{Lie}\, G
\end{tikzcd}$$

$\uparrow$ identification from last time.

Proof: Suppose D is left-invariant. Then

$$D \underset{\text{(coassoc.)}}{=} (id\otimes\varepsilon)\circ\Delta\circ D \underset{\text{(left-invariance)}}{=} (id\otimes\varepsilon)\circ(id\otimes D)\circ\Delta = (id\otimes(\varepsilon\circ D))\circ D$$

hence D is determined by $\varepsilon\circ D$. Conversely, given $d\in \mathrm{Hom}(k[G],k)$ one checks that $D := (id\otimes d)\circ\Delta$ is left invariant:



It is routine to check that λ matches derivations on both sides. $\square$

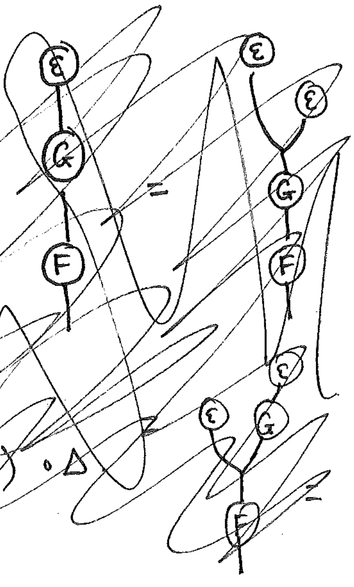
~~Lemma:~~ Define a product on $\mathrm{Hom}(k[G],k)$ via

$$f*g := (f\otimes g)\circ\Delta.$$

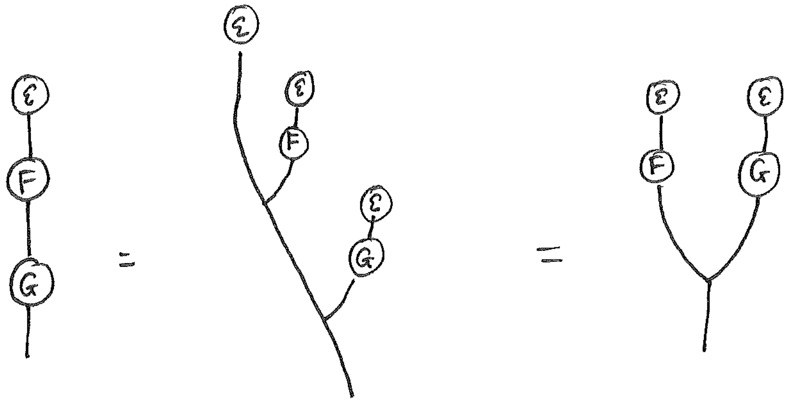
Lemma: $\lambda(F\circ G) = \lambda(F)*\lambda(G)$.

~~Proof:~~ $\cancel{\varepsilon\circ(F\circ G) = (\varepsilon\circ F)*(\varepsilon\circ G) = ((\varepsilon\circ F)\otimes(\varepsilon\circ G))\circ\Delta}$

$\cancel{= ((\varepsilon\circ F)\otimes id)\circ(id\otimes(\varepsilon\circ G))\circ\Delta}$



Proof:

$\lambda(F\circ G) =$  $= \lambda(F) * \lambda(G)$ $\square$

Important easy exercises:

ⓐ If D, D' are derivations, then so is $[D, D'] = DD' - D'D$.

ⓑ If char $k = p$ and D is a derivation, then so is D^p.

We define the Lie bracket to be the commutator of derivations under the above identifications. In this way, Lie G becomes a Lie algebra. If char $k = p$ then Lie G is a p-Lie algebra

(~~homomorphism~~ map $X \mapsto X^{[p]}$ s.t.

① $\mathrm{ad}(X^{[p]}) = (\mathrm{ad}\, X)^p$ (where $\mathrm{ad}\, X = [X, \cdot] : \mathfrak{g} \to \mathfrak{g}$)

② $(\lambda X)^{[p]} = \lambda^p X^{[p]}$ ("Frobenius linear")

③ $(X+Y)^{[p]} = (X^{[p]} + Y^{[p]} + \dots)$ terms only involving bracket.

Proposition: Any homomorphism $H \xrightarrow{\phi} G$ of group schemes induces an morphism ~~ad~~ϕ: $\mathrm{Left}(k[H]) \to \mathrm{Left}(k[G])$. We have $\phi(f * g) = \phi(f) * \phi(g)$. In particular, we have a homomorphism $d\phi : \mathrm{Lie}\, H \to \mathrm{Lie}\, G$ of Lie algebras.

Exercise: Show that the inclusion $\alpha_p \hookrightarrow \mathbb{G}_a$ induces an isomorphism $\mathrm{Lie}\,\alpha_p \xrightarrow{\sim} \mathrm{Lie}\,\mathbb{G}_a$.

Examples of p-Lie algebra structure: $\mathbb{G}_m$: $\left(x\frac{\partial}{\partial x}\right)^p = x\frac{\partial}{\partial x}$

$\mathbb{G}_a$: $\left(\frac{\partial}{\partial x}\right)^p = 0$.

Basic calculation:

Lie $Gl_n = \mathfrak{gl}_n := Mat_{n\times n}(k)$. Under this identification $[A,B] = AB - BA$.
Moreover, if char $k = p$ then $A^{[p]} = A^p$ (p^{th} power of a matrix.)
This is an important exercise! Try to do it!

Consequences: ① If $G \hookrightarrow Gl_n$ is a closed subgroup scheme then
Lie $G \subset \mathfrak{gl}_n$ is a Lie subalgebra.

② Any representation $\rho : G \to GL(V)$ gives rise to a representation
$d\rho : \mathrm{Lie}\, G \to \mathfrak{gl}(V)$. If char k is $= p$ then
this representation is restricted (i.e. $d\rho(x^{[p]}) = (d\rho(x))^{[p]}$
$= d\rho(x)^p$.)

Lie $\mathbb{G}_a$ = Lie $\mathbb{G}_m$

Exercises: Assume char $k = 0$:

① Rep $\mathbb{G}_a = \left\{ (V,\phi) \;\middle|\; \begin{matrix}\phi \in End(V) \\ \text{s.t. } \phi \text{ is nilpotent}\end{matrix} \right\} \xrightarrow{d}$ Rep Lie $\mathbb{G}_a = \{(V,\phi) \mid \phi \in End(V)\}$.

② Rep $\mathbb{G}_m = \{V \mid \text{graded vector spaces}\} \xrightarrow{d}$ Rep Lie $\mathbb{G}_m = \{(V,\phi) \mid \phi \in End(V)\}$.
(image, those $\phi \in End(V)$ which are diagonalisable with eigenvalues $\subset \mathbb{Z} \subset k$ }.
(both are fully-faithful, this is a general fact).

Assume char $k = p > 0$.

③ Rep $\mathbb{G}_a = \{ (V, \Phi_i)_{i \geq 0} \mid \dots \} \to$ Rep^{res} $\mathbb{G}_a = \{(V,\phi) \mid \phi^p = 0\}$.
$\qquad\qquad\qquad\qquad \longrightarrow (V, \Phi_1)$

Rep $\mathbb{G}_m = \{\text{graded vector spaces}\} \longrightarrow$ Rep^{res} $\mathbb{G}_m \mid (V,\phi) \mid \phi^p = \phi\}$
$\cong$
$\longrightarrow \mathbb{Z}/p\mathbb{Z}$- graded vector spaces

(neither fully-faithful).

<u>Thm</u>: Suppose that char $k = 0$. If G is connected and of finite type then

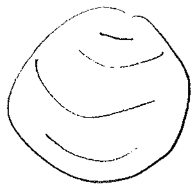
$$\begin{array}{ccc} \mathrm{Rep}\, G & \longrightarrow & \mathrm{Rep}\, \mathrm{Lie}\, G \\ \Psi & & \Psi \\ S & \longmapsto & dS \end{array}$$

is fully-faithful.

The above theorem fails in char. $p > 0$, as the above examples show.

We now define a new object which "sees everything".

<u>Motivation</u>: Consider $SO_3 = \{ A \in Gl_3 \mid AA^t = 1,\ \det A = 1 \}$.

$$SO_3 \circlearrowright S^2 =$$  .

On S^2 there are no non-zero invariant vector-fields however the differential operator $\Delta = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}$ is invariant.

The ~~eg~~ eigenspaces for Δ yield all irreducible reps of SO_3.

"spherical harmonics"

On the circle one can consider the S^1-invariant vector field $z\frac{\partial}{\partial z}$. ~~Of course~~ Eigenspaces for $z\frac{\partial}{\partial z}$ gives the theory of Fourier series: $z\frac{\partial}{\partial z}(z^m) = m z^m$.