<u>Motivation:</u> ① Consider $S^1 = \{ \overset{\in \mathbb{C}}{z} \mid \|z\| = 1 \}$.

Joe's talk: $L^2(S^1, \mathbb{C}) \cong \widehat{\bigoplus} \mathbb{C} z^m$ (Fourier series).

$z \mapsto z^m$ are the irreducible reps of $S^1$.

We can also understand this via the Lie algebra:

we have an invariant vector field $\overset{D}{\overset{\shortparallel}{z\frac{\partial}{\partial z}}}$ which.

$S^1$ preserves the eigenspaces for $D$. Note:

$D(z^m) = m z^m$, hence the eigenspaces for the vector field give the irreps. of $S^1$.

② Consider $SO_3 = \{ A \in GL_3(\mathbb{R}) \mid AA^t = 1,\ \det A = 1 \}$ "special orthogonal group"

Then $SO_3 \circlearrowright S^2$ by rotations. There are no non-zero vector fields on $S^2$ (any vector field has a zero!).

However there is a differential operator $\Delta_{S^2}$ "special Laplacian",

the eigenspace decomposition gives an ~~decom~~ isomorphism

$$L^2(S^2) = \widehat{\bigoplus_{\ell}} H_\ell ,$$

and moreover all irreducible representations of $SO_3$ occur.

The elements of $H_\ell$ are spherical Harmonics.

In this case, $\Delta_{S^2}$ arises from a differential operator "the Casimir" on $SO_3$ which is left and right invariant.

## Differential operators on schemes:

Let $X$ be an affine-scheme

An operator $D: k[X] \to k[X]$ is said to be a differential operator of order $\leq n$ if the operator

$$f \mapsto D(gf) - gD(f)$$

is of order $\leq n-1$ for all $g \in k[X]$. Of order $-1$ if $D=0$. We denote by $\mathrm{Diff}^{\leq n}(X)$ the set of differential operators of order $\leq n$.

$$\mathrm{Diff}(X) := \bigcup_{n \geq 0} \mathrm{Diff}^{\leq n}(X)$$ "algebra of differential operators"

Exercises: ① $\mathrm{Diff}^{\leq 0}(X) = k[X]$ (multiplication operators)

② Any $D \in \mathrm{Diff}^{\leq 1}(X)$ is a sum $D = f_0 + D'$ with $f \in k[X]$ and $D' \in \mathrm{Der}_k(k[X], k[X])$ (i.e. a vector field).

③ Consider $X = \mathbb{A}^1$, $k[X] = k[x]$.

Show that $\mathrm{Diff}(X) = \langle x, \frac{\partial}{\partial x} \mid [\frac{\partial}{\partial x}, x] = 1 \rangle$ (first Weyl algebra)

if $\mathrm{char}\, k = 0$.

④ The above makes sense over $\mathbb{Z}$. Show that, if $X = \mathrm{Spec}\, \mathbb{Z}[X]$

then $\mathrm{Diff}\, k(X) = \langle X, \frac{1}{n!}\frac{\partial^n}{\partial x^n} \rangle$.

E.g. $$\frac{1}{2}\frac{\partial^2}{\partial x^2}(x^i) = \begin{cases} 0 & \text{if } i \leq 1 \\ \frac{i!}{2} & \text{if } i \geq 2, \end{cases}$$ so $\frac{\partial^2}{\partial x^2}(x^2) = x$.

In fact: if $k$ is of positive characteristic

$\Rightarrow$ $\mathrm{Diff}(X_k) = \mathrm{Diff}(X) \otimes_{\mathbb{Z}} k$. (not finitely generated).

## Distributions:

Let $G$ be a group scheme. A <u>distribution of order $\leq n$</u> is a linear map $\psi : k[G] \to k$ s.t. for all $g \in k[G]$ the map

$$f \mapsto \psi(fg) - \varepsilon(f)\psi(g) \quad \text{is a distribution of order } \leq n-1.$$

(By def. the zero map is the only distribution of order $\leq -1$).

$$\text{Examples:} \quad \mathrm{Dist}^{\leq 0}(G) = k\varepsilon, \quad \mathrm{Dist}^{\leq 1}(G) \overset{\text{Hom}(k \oplus I/I^2,\, k)}{\overset{\shortparallel}{=}} k\varepsilon \oplus \mathrm{Lie}\, G.$$

~~Lemma~~

**Lemma:** $\mathrm{Dist}^{\leq n} = \left(k[G]/(I^{n+1})\right)^*$.

Explains the name distribution: functionals which are zero away from $e$ id $\in G$.

We have a commutative diagram of isomorphisms:

$$\begin{tikzcd}
\mathrm{Left}(X) \arrow[r, "D \mapsto \varepsilon \circ D"] & \mathrm{Hom}(k[G], k) \\
\text{left-invariant differential operators on } X \arrow[u, phantom, "\cup"] \arrow[r] & \mathrm{Dist}(G) \arrow[u, phantom, "\cup"] \\
\text{left-invariant vector fields on } X \arrow[u, phantom, "\cup"] \arrow[r] & \mathrm{Der}(G) = \mathrm{Lie}(G) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

Do $G_a$ example ~~next~~ here.

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$$\mathrm{Dist}_{\leq n}(G) = \mathrm{Hom}(k[G]/(I^{n+1}), k) \subset \mathrm{Hom}(k[G], k)$$

**Claim:** $f \in \mathrm{Dist}_{\leq n}(G)$, $g \in \mathrm{Dist}_{\leq m}(G)$, $\underset{(f \otimes g) \circ \Delta}{\overset{\shortparallel}{f * g}} \in \mathrm{Dist}_{\leq n+m}(G)$.

Proof (Use that $\Delta(I) \subset I \otimes k[G] + k[G] \otimes I$.)

$$\mathrm{Dist}(G) := \bigcup_{n \geq 0} \mathrm{Dist}_{\leq n}(G).$$

## Distributions and enveloping algebras:

Recall: $V$ v.s. $/k$ $\qquad T(V) = \bigoplus V^{\otimes n}$ tensor algebra of $V$.

$$S(V) = T(V)/(v\otimes w - w\otimes v) \quad \text{symmetric algebra of } V.$$

( $\mathrm{Sym}_n \circlearrowright V^{\otimes n}$ for all $n$ and we are taking the coinvariants )

$$= \bigoplus (V^{\otimes n})_{\mathrm{Sym}_n}$$

$$\Gamma(V) = \bigoplus (V^{\otimes n})^{\mathrm{Sym}_n} = \bigoplus \Gamma^i(V)$$

divided power algebra of $V$.

One needs to be careful in defining multiplication,

$v \in \Gamma^i(V)$, $w \in \Gamma^j(V)$ $\qquad v \bullet w := \sum_{\sigma \in \mathrm{Sym}_{i+j}/\mathrm{Sym}_i \times \mathrm{Sym}_j} \sigma(v\otimes w).$

Exercise: ① Suppose $V$ ~~has a basis~~ is one-dimensional. Then

$$\Gamma(V) = \bigoplus k x^{(i)} \quad \text{where} \quad x^{(i)} x^{(j)} = \binom{i+j}{i} x^{(i+j)}.$$

② Describe $\Gamma(V)$ for $V$ a finite-dimensional vector space.

③ Show that one has a canonical isomorphism $(S^i(V))^* \xrightarrow{\sim} \Gamma^i(V^*)$.

Now suppose $\mathfrak{g}/k$ is a Lie algebra. We define

$$U(\mathfrak{g}) = T(\mathfrak{g})/(v\otimes w - w\otimes v - [v,w],\ \forall v,w \in \mathfrak{g}) \quad \text{"universal enveloping algebra"}$$

$$U^0(\mathfrak{g}) = U(\mathfrak{g})/(x^p - x^{[p]}) \quad (\text{if } \mathfrak{g} \text{ is restricted}). \quad \text{"restricted universal enveloping algebra".}$$

Poincaré-Birkhoff-Witt theorem: The sets $\{x_1^{i_1} \dots x_h^{i_h} \mid 0 \le i_j \le \infty\}$

$\{x_1^{i_1} \dots x_h^{i_h} \mid 0 \le i_j < p\}$

are bases for $U(\mathfrak{g})$ and $U^0(\mathfrak{g})$ resp.

The map $\mathrm{Lie}\, G \longrightarrow \mathrm{Dist}(G)$ induces a map

$$U(\mathrm{Lie}\, G) \longrightarrow \mathrm{Dist}(G)$$ which is an iso. in char $0$ and induces an embedding $U^{0}(\mathrm{Lie}\, G) \longrightarrow \mathrm{Dist}(G)$ in char $p$.

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Given any $G$-module $M$ we may define a $\mathrm{Dist}(G)$-module structure

via
$$\begin{tikzcd} M \arrow[r, "c"] \arrow[rr, bend right, "x\cdot"'] & M \otimes k(G) \arrow[r, "\mathrm{id}\otimes x"] & M \end{tikzcd}.$$

<u>Thm</u>: If $G$ is connected the above functor

$$\mathrm{Rep}\, G \longrightarrow \mathrm{Dist}\, G\text{-mod}$$

is fully-faithful.

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Examples of Distribution Algebras:

$\underline{\underline{G_a}}$ : Let $\phi_i : k[x] \longrightarrow k$ be given by $\phi_i(x^j) = \delta_{ij}$.

Then $(\phi_i * \phi_j)(x^k) = (\phi_i \otimes \phi_j)\left( \sum \binom{k}{l} x^{k-l} \otimes x^l \right)$

$$= \begin{cases} 0 & \text{if } i+j \neq l \\ \binom{k}{i} & \text{if } i+j = l. \end{cases}$$

Hence $\phi_i * \phi_j = \binom{i+j}{i} \phi_{i+j}$.

Hence $\operatorname{Dist} G_a \cong \Gamma(k)$.

$\underline{\underline{G_m}}$ : Here the answer is somewhat complicated, we simply state it.

Consider the algebra $\mathbb{Q}[x]$

$$A = \left\{ f \in \mathbb{Q}[x] \;\middle|\; \begin{array}{l} f \text{ takes integral values} \\ \text{on } \mathbb{Z} \subset \mathbb{Q} \end{array} \right\}.$$

__Lemma:__ $A$ is free over $\mathbb{Z}$ with basis $\left\{ \binom{x}{i} := \dfrac{x(x-1)\dots(x-i+1)}{i(i-1)\dots 1} \;\middle|\; i \in \mathbb{Z}_{\geq 0} \right\}$.

For $f \in A$, define $\phi_f(x^k) = f(k)$. This gives a map

$$\phi : A \otimes_{\mathbb{Z}} k \longrightarrow \operatorname{Dist}(G_m).$$

__Lemma:__ This is an isomorphism.

## Review of the rep theory of $\mathfrak{sl}_2(\mathbb{C})$

$\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e$ where $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f.$

Let $V$ denote an $\mathfrak{sl}_2(\mathbb{C})$-module. We set

$$V_\lambda = \{ v \in V \mid h.v = \lambda v \}.$$

<u>Thm:</u> If $V$ is a finite dim $\mathfrak{sl}_2(\mathbb{C})$-module then

① $h$ acts semi-simply on $V$ and all eigenvalues belong to $\mathbb{Z} \subset \mathbb{C}$. Hence

$$V = \bigoplus_{\lambda \in \mathbb{Z}} V_\lambda$$

② If $V$ is simple and $n$ is maximal with $V_n \neq 0$ then $V$ is generated by $V_n$ and the weight spaces of $V$ are all one-dimensional and of the form $n, n-2, n-4, \dots, -n$. Any two $V', V$ satisfying these properties are isomorphic.

③ Any $V$ is semi-simple.

$[e,f] = h.$

$ef - fe$

$2(n-1) - n = n-2.$

In fact, any such $V$ has the following form:

$$\begin{tikzcd}
\mathbb{C}v_{-n} \arrow[r, bend right, "1"'] & \mathbb{C}v_{-n+2} \arrow[l, bend right, "n"'] \arrow[r, bend right, "2"'] & \cdots \arrow[l, bend right] & \cdots \arrow[r, bend right] & \mathbb{C}v_{n-4} \arrow[l, bend right, "3"'] \arrow[r, bend right, "n-1"'] & \mathbb{C}v_{n-2} \arrow[l, bend right, "2"'] \arrow[r, bend right, "n"'] & \mathbb{C}v_n \arrow[l, bend right, "1"']
\end{tikzcd}$$

$= ef$

$= e$

$\circlearrowleft_{-n} \quad \circlearrowleft_{-n+2} \quad \circlearrowleft_{n-4} \quad \circlearrowleft_{n-2} \quad \circlearrowleft_{n}$