Frobenius / Frobenii

Suppose that X is ~~given~~ an algebraic subset of $(\overline{\mathbb{F}}_q)^n$ which is "defined over $\mathbb{F}_q$" (i.e. $X = \{ x \mid f_i(x) = 0 \}$ for some polynomials

$$f_i \in \mathbb{F}_q[x_1, \dots, x_n]).$$

Then if $(\lambda_1, \dots, \lambda_n) \in X$ then so is $(\lambda_1^q, \dots, \lambda_n^q) \in X$.

(More generally, this would work for any $\sigma \in \mathrm{Gal}(\overline{K}/K)$ if X is defined over K.)

Big difference: here Frobenius is algebraic (defined by polynomials).

Frobenius plays an important role in the study of finite groups of Lie type. For example $GL_n(\mathbb{F}_q)$ has an automorphism given by $x_{ij} \mapsto x_{ij}^p$.

Formally, for any $\mathbb{F}_q$-algebra A the Frobenius on A is the map $a \mapsto a^q$. Suppose that G is a groupscheme defined over $\mathbb{F}_q$. Then the Frobenius on $k(G)$ induces a morphism of Hopf algebras ($\Delta(x)^p = \Delta(x^p)$) and hence we obtain a ~~map~~ morphism of algebraic groups:

$$Fr : G \longrightarrow G.$$

~~Thus~~ Precomposition with Fr defines a map

$$\mathrm{Rep}\, G \longrightarrow \mathrm{Rep}\, G \qquad \text{"Frobenius twist".}$$
$$V \longmapsto V^{Fr}$$

In terms of coalgebras this is the map that sends M to the comodule

with coaction: $M \longrightarrow M \otimes k[G] \xrightarrow{id \otimes Fr} M \otimes kG$

<u>Example:</u> Recall that Rep $\mathbb{G}_m$ is semi-simple and that the simple modules are given by $z \mapsto z^m$ (e.g. $M = ke$ with coaction $e \mapsto e \otimes t^m$).
parametrized by $m \in \mathbb{Z}$

The Frobenius twist sends $z \mapsto z^m$ to $z \mapsto z^{pm}$. (i.e. $e \mapsto e \otimes t^{pm}$).

That is, it "stretches out the grading by p".

<u>Exercise:</u> ① Show that the action of Lie G on V^{Fr} is trivial.

② Frobenius induces a morphism Dist $G \to$ Dist $\mathbb{G}_a$. Show that Dist $\mathbb{G}_a$

$$\text{Dist } \mathbb{G}_a = k[\underset{x,\, x^{(p)},\, x^{(p^2)}, \ldots}{\Phi_1, \Phi_2, \ldots}],$$ sends $\Phi_2 \to \Phi_1, \ldots$ (Perhaps better to rephrase this in terms of divided powers.)

<u>Frobenius kernels:</u> G is a group scheme over $\mathbb{F}_q = k$.

For any $\mathbb{F}_q$-algebra A, $Fr: G(A) \longrightarrow G(A)$ Frobenius morphism.

$$\begin{matrix} G(A) & \longrightarrow & G(A) \\ \| & & \| \\ \mathrm{Hom}(k[G], A) & & \mathrm{Hom}(k[G], A) \end{matrix}$$

<u>Note:</u> Fr is injective for A a field but not in general.

We define $G_r := \ker Fr^r$.

Some work shows: G_r is representable by the ~~k[G]-alg~~

$$k[G_r] = \mathrm{Spec}\; k[G]/((I)^{p^r}).$$

<u>E.g.</u> $\mathrm{Spec}\,(\mathbb{G}_a)_r = k[x]/(x^{p^r})$, $\mathrm{Spec}(\mathbb{G}_m)_r = k[x]/(x^{p^r}-1) = k[x]/(x-1)^{p^r}$.

Fact: $\mathrm{Dist}\, G_1 = U^0(\mathrm{Lie}\, G)$.

One can imagine one has a short exact sequence of group schemes

$$\{1\} \longrightarrow G_1 \longrightarrow G \xrightarrow{Fr} G \longrightarrow \{1\}.$$

(This "explains" the exercise above about trivial Lie G action.)

Dist $\mathbb{G}_m$: Recall $k[\mathbb{G}_m] = k[x^{\pm 1}]$ with $\Delta(x) = x \otimes x$, $I = (x-1)$.

Define $\delta_i((x-1)^j) = \delta_{i,j}$. Then

$$\delta_i(x^l) = \delta_i(((x-1)+1)^l)$$
$$= \delta_i\Big(\sum_{j=0}^{l} \binom{l}{j}(x-1)^j\Big) = \binom{l}{i}.$$

In other words, δ_i sends $x^l \mapsto \binom{l}{i}$. Clearly $\delta_0, \delta_1, \dots, \delta_m$ give a k-basis for $k[\mathbb{G}_m]/(I)^m$. Hence $\delta_0, \delta_1, \dots$ give a k-basis for $\mathrm{Dist}(\mathbb{G}_m)$. Hence structure constants are given by

$$(\delta_i * \delta_j)(x^l) = (\delta_i \otimes \delta_j)(x^l \otimes x^l) = \binom{l}{i}\binom{l}{j}$$

structure constants of multiplication of binomial coefficients.

Exercise:

$$\binom{l}{r}\binom{l}{s} = \sum_{i=0}^{\min(r,s)} \binom{r+s-i}{r-i,\, s-i,\, i}\binom{l}{r+s-i}$$

Hence:

$$\delta_r * \delta_s = \sum (\ \ )\, \delta_{r+s-i}.$$

In particular: $\delta_1 * \delta_r = \binom{1+r}{1,r,0}\delta_{1+r} + \binom{1+r-1}{r-1,\cancel{0},1}\delta_r$

$$= (r+1)\delta_{1+r} + r\delta_r .$$

$\Longrightarrow \delta_r = \dfrac{\delta_1 - (r-1)}{r} * \delta_{r-1}$

Thus: $(\delta_1 - r) * \delta_r = (r+1)\delta_{r+1}$.

Hence in char 0: $\delta_{r+1} = \binom{\delta_1}{r+1} = \dfrac{\delta_1(\delta_1-1)\dots(\delta_1-r)}{(r+1)!}$ $\qquad \delta_r = \binom{\delta_1}{r}$.

Thus $\mathrm{Dist}\, \mathbb{G}_m \cong k[\delta_1]$.

Fun exercise: Consider: $A = \{ f : \mathbb{Z} \to \mathbb{Z} \mid$ f is given by a polynomial in $\mathbb{Q}$ with coefficients $\}$.

① Show that A has a free $\mathbb{Z}$-basis given by $n \mapsto \binom{n}{i}$ for $i = 0, 1, \dots$
(Thus any $\mathbb{Z}$-valued polynomial is expressible via binomial coefficients).

② for $\phi : \mathbb{Z} \to \mathbb{Z}$ arbitrary, show that ϕ is given by a polynomial in $\mathbb{Q}$ if and only if the map $\mathbb{Z}[x^{\pm 1}] \to \mathbb{Z}$ given by $x^i \mapsto \phi(i)$ vanishes on the ideal $(x-1)^l$ for some $l \geq 0$.

③ Deduce that $\mathrm{Dist}\, \mathbb{G}_m = k \otimes A$, for any k.

④ ~~Show that Hom(~~ ~~Dist~~ ~~$\mathbb{G}_m$~~

Question: What are the irreducible reps of $\mathrm{Dist}\, \mathbb{G}_m$ in char p?

(Hint: $k = \mathbb{F}_p$ seems to give $\mathbb{Z}_p$, p-adic integers.)

Kostant $\mathbb{Z}$-form

$$sl_2 = sl_2(\mathbb{C}) = \underset{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}{\underset{\shortparallel}{\mathbb{C}f}} \oplus \underset{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{\underset{\shortparallel}{\mathbb{C}h}} \oplus \underset{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}{\underset{\shortparallel}{\mathbb{C}e}}.$$

$U = U(sl_2)$ = universal enveloping algebra.

The Kostant $\mathbb{Z}$-form $U_{\mathbb{Z}}$ of U is the subalgebra of sl_2 generated by the elements

$$\underset{i \geq 0}{f^{(i)} := \frac{f^i}{i!}}, \quad \underset{j \geq 0}{\binom{h}{j} := \frac{h(h-1)\dots(h-j+1)}{j!}}, \quad \underset{i \geq 0}{e^{(i)} := \frac{e^i}{i!}}.$$

Thm: $U_{\mathbb{Z}}$ has a $\mathbb{Z}$-basis given by $\left\{ f^{(i)} \binom{h}{j} e^{(l)} \,\middle|\, i, j, l \geq 0 \right\}$.

In particular, $U_{\mathbb{Z}}$ is a $\mathbb{Z}$-lattice inside U.

Idea of proof: One needs to show that any product

$$f^{(i_1)} \binom{h}{j_1} e^{(l_1)} f^{(i_2)} \binom{h}{j_2} e^{(l_2)}$$

is an integral linear combination of ~~bas~~ (purported) basis elements:

$$f^{(i_1)} f^{(i_2)} = \binom{i_1 + i_2}{i_1} f^{(i_1+i_2)}, \quad e^{(l_1)} e^{(l_2)} = \dots$$

$$\binom{h}{j_1}\binom{h}{j_2} = \text{as earlier.}$$

$$(h-2)e = eh \implies e^{(i)} \binom{h}{j} = \binom{h-2i}{j} e^{(i)} \qquad (*)$$

$$(h+2)f = fh \implies f^{(i)} \binom{h}{j} = \binom{h+2i}{j} f^{(i)}. \qquad (*)$$

Key relation:

$$e^{(m)} f^{(n)} = \sum_{j=0}^{\min(m,n)} f^{(n-j)} \binom{h-m-n+2j}{j} e^{(m-j)} .$$

Exercise: Establish this relation by first proving

$$e f^{(n)} = f^{(n)} e + f^{(m-1)} (h-m+1).$$

(Is a consequence of a relation on the sl_2-sheet from last time)

Theorem (~~Haboush~~) $\quad \mathrm{Dist}\, SL_2 \cong U \underset{\mathbb{Z}}{\otimes} k = U^-_{\mathbb{Z}} \otimes U^0_{\mathbb{Z}} \otimes U^+_{\mathbb{Z}}$.

Idea of proof:

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \times \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \times \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \longrightarrow SL_2$$

multiplication: $\alpha \in \mathbb{G}_a \quad \lambda \in \mathbb{G}_m \quad \beta \in \mathbb{G}_a$

gives an open immersion which is an isomorphism of in a neighbourhood of the identity. Hence

$$\mathrm{Dist}\, \mathbb{G}_a \otimes \mathrm{Dist}\, \mathbb{G}_m \otimes \mathrm{Dist}\, \mathbb{G}_a \xrightarrow{\sim} \mathrm{Dist}\, SL_2 \quad \text{(as vector spaces)}.$$

If we work over the integers this gives us the description

of $\mathrm{Dist}\, SL_{2,\mathbb{Z}} \hookrightarrow \mathrm{Dist}\, SL_{2,\mathbb{C}} \cong U$.

$$\mathrm{Dist}\, SL_{2,\mathbb{Z}} = U_{\mathbb{Z}}$$

$\square$

~~As f.d. representations of $sl_2(\mathbb{C})$, we can draw pictures of representations of $\mathrm{Dist}\, SL_2$. Note that $U(\mathfrak{h})_{\mathbb{Z}} = \mathbb{Z}\{\binom{h}{i} \mid i \geq 0\}$ acts semisimply on any module coming from SL_2.~~

<u>Lemma:</u> Suppose that V is a finite-dimensional sl_2-module.

(1) $V = \bigoplus V_m$, $\quad V_m = \{ v \in V \mid \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \cdot v = \lambda^m v \}$.

(2) For all $v \in V_m$, $\binom{h}{i} \cdot v = \binom{m}{i} v$, for all $i \in \mathbb{Z}_{\geq 0}$. In particular, U_k^0 acts semi-simply. Moreover, $V_m = \{ v \in V \mid \binom{h}{i} \cdot v = \binom{m}{i} \cdot v \ \forall i \in \mathbb{Z}_{\geq 0} \}$.

(3) $e^{(n)}(V_m) \subset V_{m+n}$, $f^{(n)}(V_m) \subset V_{m-n}$.

<u>Proof:</u> (1) Follows because Rep $\mathbb{G}_m$ is semi-simple.

It is easy to see that $\binom{h}{i}$ acts as $\binom{m}{i}$ on V_m, in order to see the "moreover" statement we need to know

<u>Lemma:</u> If $\binom{m}{i} = \binom{m'}{i}$ mod p for all i, then $m = m'$.

<u>Proof:</u> For m positive this follows from Lucas' formula:

if $m = m_0 + pm_1 + \dots + p^l m_l$, $i = i_0 + pi_1 + \dots + p^l i_l$ denote p-adic expansions then

$$\binom{m}{i} = \binom{m_0}{i_0} \cdots \binom{m_l}{i_l} \mod p.$$

Hence $\binom{m}{p^i} = m_i$ mod p, and hence $m = m'$. (All p-adic digits agree.)

For general m use that Lucas' formula also holds for m any p-adic number. □

Finally: $(*) + (*) \Rightarrow$ (3).

The lemma implies that we can draw pictures of SL_2-modules in the same way that we depict $\mathfrak{sl}_2$-modules, except we need to keep track of all $e^{(i)}$'s, $f^{(i)}$'s not just e, f.

The natural module nat yields the following Dist SL_2-module.

<u>Example:</u>

<u>nat:</u>

$$\begin{tikzcd} y \arrow[r, bend right, "e"'] \arrow[loop below, "-1"] & x \arrow[l, bend right, "f"'] \arrow[loop below, "1"] \end{tikzcd}$$

We now consider divided and symmetric powers of nat.

<u>Notation:</u> $\boxed{x^i y^j} := \sum_{\sigma \in S_{i+j}/S_i \times S_j} \sigma(\underbrace{x \otimes \ldots \otimes x}_{i \text{ times}} \otimes \underbrace{y \otimes \ldots \otimes y}_{j \text{ times}}) \in \Gamma^{i+j}(\text{nat})$.

<u>Note that:</u> Under the natural map $\Gamma^n(\text{nat}) \to S^n(\text{nat})$, $\boxed{x^i y^{n-i}} \mapsto \binom{n}{i} x^i y^{n-i}$.

$$e(y \otimes \ldots \otimes y) = x \otimes y \otimes \ldots \otimes y + y \otimes x \otimes \ldots \otimes y + \ldots + y \otimes \ldots \otimes y \otimes x.$$

$$e(y^n) = e \cdot (y \otimes \ldots \otimes y) = n y^{n-1} x.$$

$$e(\boxed{y^n}) = e \cdot (y \otimes \ldots \otimes y) = \boxed{y^{n-1} x}.$$

$$e(\boxed{x y^{n-1}}) = \cancel{\otimes}\, 2 \boxed{x^2 y^{n-2}} \quad \text{etc.}$$

We use the following:

$$\Delta(e^{(\ell)}) = \sum_{i+j=\ell} e^{(i)} \otimes e^{(j)}$$

$$\Delta(f^{(\ell)}) = \sum_{i+j=\ell} f^{(i)} \otimes f^{(j)}$$

We arrive at the following picture:

$$\begin{tikzcd}
\Gamma^n(\mathrm{nat}) & \boxed{y^n} \arrow[r, bend right, "1"'] & \boxed{xy^{n-1}} \arrow[l, bend right, "n"'] \arrow[r, bend right, "2"'] \arrow[d, "\binom{n}{n-1}"'] & \cdots & \boxed{x^{n-1}y} \arrow[l, bend right, "2"'] \arrow[r, bend right, "n"'] \arrow[ll, bend right=50, "\binom{3}{2}"'] \arrow[d, "\binom{n}{1}"'] & \boxed{x^n} \arrow[l, bend right, "1"'] \arrow[lll, bend right=50, "\binom{2}{2}"'] \arrow[d, "\binom{n}{0}"'] \\
S^n(\mathrm{nat}) & y^n \arrow[r, bend right, "n"'] \arrow[from=u, "\binom{n}{n}"'] & xy^{n-1} \arrow[l, bend right, "1"'] \arrow[r, bend right, "n-1"'] & \cdots & x^{n-1}y \arrow[l, bend right, "n-1"'] \arrow[r, bend right, "1"'] & x^n. \arrow[l, bend right, "n"']
\end{tikzcd}$$

Theorem: ① The image of $\Gamma^n(\mathrm{nat}) \longrightarrow S^n(\mathrm{nat})$ is a simple SL_2-module L_n.

② All simple SL_2-modules are of this form.

Hence,

③ We have

$$\mathrm{ch}\, L_n = \sum_{\substack{0 \le i \le n \\ \binom{n}{i} \neq 0 \\ \mathrm{mod}\ p}} e^i .$$

In other words, if we consider

$\mathrm{ch}\, L_0$

$\mathrm{ch}\, L_1$

$\mathrm{ch}\, L_2$

$\mathrm{ch}\, L_3$

1

1 1

1 2 1

1 0 0 1

1 1 0 1 1

1 2 1 1 2

1 0 0 2 0 0

⋮

beautiful fractal triangle!