Recall that a group scheme is <u>finite</u> if $k[G]$ is finite-dimensional.

<u>infinitesimal</u> if finite and $I^m = 0$ for some m ($I = \ker \varepsilon$, aug. ideal).

$\Rightarrow$ G has a unique k'-point for any k-field k'.

<u>Rmk</u>: If G is infinitesimal, $\mathrm{Dist}\, G = \mathrm{Hom}(k[G], k) = k[G]^*$.

If A is a finite dim. Hopf algebra, so is A^* and $A \xrightarrow{\sim} (A^*)^*$.

$$A \quad \begin{array}{ccc} \text{finite dim.} & & \\ \text{comodules} & \xrightarrow{\sim} & \text{modules} \\ \text{modules} & \xrightarrow{\sim} & \text{comodules}. \end{array}$$

Hence $\mathrm{Rep}\, G = k[G]\text{-comod} \simeq \mathrm{Dist}\, G\text{-modules}$.

(Fully-faithful embedding $\mathrm{Rep}\, G \hookrightarrow \mathrm{Dist}\, G\text{-mod}$ is trivial in this case.)

Of course, G_r is infinitesimal for all $r \geq 1$.

- $\mathrm{Lie}\, G = (I/I^2)^* = \mathrm{Lie}\, G_r$ for all $r \geq 1$.
- $U^0(\mathrm{Lie} G) = \mathrm{Dist}\, G_1$.
- $\mathrm{Dist}\, G_r = (\mathrm{Dist}\, G)_{< p^r}$ for all r.

<u>Examples</u>: ① $\mathrm{Dist}\, \mathbb{G}_{a,r} = \langle x^{(i)} \mid 0 \leq i \leq p^r - 1 \rangle \subset \mathrm{Dist}\, \mathbb{G}_a$

② $\mathrm{Dist}\, \mathbb{G}_{m,r} = \left\langle \binom{x}{i} \mid 0 \leq i \leq p^r - 1 \right\rangle \subset \mathrm{Dist}\, \mathbb{G}_m$.

<u>Exercise (worthwhile)</u>: Show that Fr induces the following maps on Dist:

ⓐ $x^{(n)} \mapsto \begin{cases} x^{(n/p)} & \text{if } n = 0 \bmod p \\ 0 & \text{o/w} \end{cases}$ on $\mathbb{G}_a$.

ⓑ $\binom{x}{i} \mapsto \binom{x}{\frac{i - i_0}{p}}$ on $\mathbb{G}_m$.

What is all of this for sl_2?

Recall from last time:

$$U_{\mathbb{Z}} = \left\langle f^{(i)} := \frac{f^i}{i!},\ \binom{h}{i} = \frac{h(h-1)\cdots(h-i+1)}{i!},\ e^{(i)} = \frac{e^i}{i!} \right\rangle \subset U(sl_2(\mathbb{C}))$$

Kostant $\mathbb{Z}$-form

k a field of char p.

$U_k := U_{\mathbb{Z}} \otimes_{\mathbb{Z}} k.$ $\quad$ $\mathrm{Dist}\, SL_{2,r} = \langle f^{(i)}, \binom{h}{i}, e^{(i)} \mid 0 \le i \le p^r - 1 \rangle.$

Exercise: Check explicitly that f, h, e generate a subalgebra isomorphic to $U^0(sl_2) = U(sl_2)/(e^p = 0, f^p = 0, h^p = h)$.

PICTURE FOR SL_2, $p=2$ here.

Recall: ① $\Gamma^m(\mathrm{nat}) \twoheadrightarrow L_m \hookrightarrow S^m(\mathrm{nat})$

② $\{L_m \mid m \geq 0\}$ are the irreducible SL_2-modules.

Exercise: $e^{(n)}, f^{(n)}$ act trivially on L_m for $m \le p^r - 1$, $n \geq p^r$.

Hence: L_m for $m = 0, 1, \ldots, p^r - 1$ restrict to simple $SL_{2,r}$-modules.

Thm: L_m, $m = 0, 1, \ldots, p^r - 1$ are the simple $SL_{2,r}$-modules.

Corollary: Every simple $SL_{2,r}$-module extends to an SL_2-module.

Clifford Theory:

G finite group.

N normal subgroup.

V an irred. f.d. G-rep. Let $L \subset V|_N$ be a simple submodule.

Then obviously:

① For any $g \in G$, $v \in L$ $\quad n \cdot g^{-1}v = g^{-1} \cdot gng^{-1} \cdot v$

$$\Rightarrow g^{-1}L \cong L^g \text{ ~~twisted~~ } (L \text{ twisted by automorphism } n \mapsto gng^{-1}).$$

② $$V \cong \sum_{g \in G/N} g^{-1} L.$$

Hence V is semi-simple as an N-module, and all simple N-modules that occur in V form one orbit of G conjugation action on $\operatorname{Irr} N$!

This basic observation is the heart of Clifford theory.

Suppose further that:

for any N-module (irreducible) L,

(*) $L^g \cong L$. $\quad$ (Action is trivial.)

Then $V|_N \xrightarrow{\sim} L \oplus L \oplus \dots \oplus L$ and hence

$$V \xrightarrow{\sim} L \otimes \operatorname{Hom}_N(L, V).$$

$$l \otimes \varphi \mapsto \varphi(l).$$

(As N-modules)

$$\boxed{gl \otimes \varphi^g = \varphi^g(gl) = g\varphi(g^{-1} \cdot gl) = g\varphi(l).}$$

Moreover, this is an isomorphism of G-modules if L extends to an G-module !!

Note that the N-action on $\operatorname{Hom}_N(L,V)$ is trivial, in particular, $\operatorname{Hom}_N(L,V)$ ~~is~~ a G/N-module comes from

Roughly one can think that in this situation the exact sequence

$$N \hookrightarrow G \to G/N$$

looks something like a direct product, and we can produce simple modules for G by "tensoring" simple modules for N and G/N.

For SL_2 (and more generally any reductive group split $/\mathbb{F}_p$) all of the above miracles happen for

$$SL_{2,r} \hookrightarrow SL_2 \xrightarrow{(Fr)^r} SL_2$$

- SL_2 acts trivially on simple $SL_{2,r}$ modules (they are all of different dimensions). conjugating
- any simple $SL_{2,r}$-module extends to an SL_2-module (we have seen why above).

"Clifford theory for SL_2":

~~Thm~~ Write $m = m_0 + p^r m_1$ with $0 \le m_0 \le p^r$. Then

$$L_m \cong L_{m_0} \otimes (L_{m_1})^{(r)}.$$

Corollary: ("Steinbergs tensor product thm" for SL_2:)

For any $m \geq 0$, write $m = m_0 + pm_1 + p^2 m_2 + \dots + p^\ell m_\ell$ $\quad 0 \le m_i \le p-1$. Then

$$L_m \cong L_{m_0} \otimes (L_{m_1})^{(1)}_{Fr} \otimes \dots \otimes (L_{m_\ell})^{(\ell)}.$$

Example: $p=3$.