Back to the case of GL_n. (For concreteness, theory works for any totally disconnected group). If $K_0, K_1, \ldots$ denote our sequence of open nbhds of the identity we can consider the chain

$$V^{K_0} \subset V^{K_1} \subset V^{K_2} \subset \ldots$$

smooth $\Rightarrow$ every vector lives in V^{U} for some $U < G$ open, hence in some V^{K_i}. filtration is exhaustive.

admissible $\Rightarrow$ each V^{K_i} is finite-dimensional.

K_0 action on V^{K_i} factors over $K_0/K_i \cong GL_n(\mathcal{O}_K/\mathfrak{m}_K^i)$

$\uparrow$ finite group e.g. $GL_n(\mathbb{Z}/p^m\mathbb{Z})$.

KEY POINT: $K_i \subset K_0$ is normal!

Hence: $$V^{K_i} = \bigoplus_{\rho \in \widehat{K_0/K_i}} \underbrace{V^{K_i}(\rho)}_{\rho \text{ isotypic component.}}$$

Passing to the limit: $$V = \bigoplus_{\rho \in \widehat{K_0}} \underbrace{V(\rho)}_{\rho \text{ isotypic component}}$$

Lemma: V admissible $\iff$ each $V(\rho)$ is finite-dimensional.

Proof: If $V(\rho)$ is infinite-dimensional, then for some ρ $\ker\rho$ contains a K_i for some i, then $V(\rho) \subset V^{K_i}$ is ∞-dimensional, hence V is not smooth.

For the other direction, note that ρ can occur in at most one of the reps

V^{K_0} in V^{K_1}/V^{K_0} V^{K_2}/V^{K_1} V^{K_3}/V^{K_2} ...

indeed, if $\rho \in V^{K_{i+1}}/V^{K_i}$ then ρ has K_{i+1} invariants, but no K_i invariants. Because each of these spaces is finite dimensional, the result follows. $\square$

Rmk: This is like the theory of K-finite vectors in the rep. theory of real Lie groups. A big difference is that here we have no idea about the rep theory of e.g. $GL_{10}(\mathbb{Z}/p^n\mathbb{Z})$ for large n, whereas we know the rep. theory of compact Lie groups rather well.

Examples: ① $\mathcal{F} = \{\varphi : \mathbb{Z}_p \to \mathbb{C} \mid \varphi \text{ loc. constant}\}$, $\mathcal{F}^{p^m\mathbb{Z}_p} = \{\varphi \mid \varphi \text{ constant on } p^m\mathbb{Z}_p\text{-orbits}\}$

$= \text{reg rep. of } \mathbb{Z}_p/p^m\mathbb{Z}_p$.

Hence $$\mathcal{F} = \bigoplus_{\substack{\chi : \mathbb{Z}_p \to \mathbb{C}^\times \\ \text{cts}}} \mathbb{C}_\chi$$

② $I = \{\varphi : \mathbb{P}^1(K) \to \mathbb{C} \mid \text{loc. constant}\}$. $\mathcal{F}^{K_n} = \{\varphi : \mathbb{P}^1(\mathcal{O}_K/\mathfrak{m}_K^n) \to \mathbb{C}\}$

$$\mathcal{F} = \varinjlim \mathbb{C}[\mathbb{P}^1(\mathcal{O}_K/\mathfrak{m}_K^n)].$$

Let V be a rep. of $Gl_n(K)$.

$\xi : V \to \mathbb{C}$ is smooth if $\mathrm{Stab}_G\, \xi$ is open.

$\hat{V} = \{\text{smooth vectors } \xi : V \to \mathbb{C}\}$ "smooth dual"

Lemma: Assume V is a smooth rep. of $Gl_n(K)$.
If $V = \bigoplus_{\mathfrak{z} \in \hat{K}_0} V(\mathfrak{z})$ then $\hat{V} = \bigoplus_{\mathfrak{z} \in \hat{K}_0} V(\mathfrak{z})^*$.

In particular, if V is smooth, admissible then so is $\hat{V}$, and $V \xrightarrow{\sim} \widehat{(\hat{V})}$.

Proof: $\xi : V \to \mathbb{C}$ is smooth if and only if
ξ vanishes on all but finitely many $V(\mathfrak{z})$,
and the lemma follows.

We will now attempt to give a bird's eye view of the smooth admissible reps. of $GL_1(K)$ and $Gl_2(K)$.

First we need a digression on norms. (INSERT SECTION ON CANONICAL NORMS HERE!)

Case of $GL_1(K) = K^\times$: Let V be a smooth admissible rep of $K^\times$.

Note: $K^\times$ is abelian, so $K_j := 1 + m_K^j \subset \mathcal{O}_K^\times$ is normal.

smooth admissible: $V = \bigcup V^{K_j}$ $\leftarrow$ f.d.

$K^\times / K_j \cong \mathbb{Z} \times (\mathcal{O}_K / m_K^j)^\times \curvearrowright V^{K_j}$.

Hence if V is irreducible, V is one-dimensional and V is determined by a character $|\cdot|^c \chi(\,)$

$\chi : \mathcal{O}_K^\times$ cts ch., $c \in \mathbb{C}$.

(LATER)

Canonical norms:

For the product formula to hold we define: $\|x\|_v = (\#\mathcal{O}_K/\mathfrak{p})^{-\mathrm{val}_{\mathfrak{p}} x}$ finite places

$\|x\|_v = |i(x)|$ real places

$\|x\|_v = |i(x)|^2$ complex places.

Eg: For $K = \mathbb{Q}_p$, $|p| = \varepsilon$ for any $0 < \varepsilon < 1$ would do... why $1/p$?
In some sense this is justified by the product formula, but is still a little mysterious.

Tate's observation: K_v completion is locally compact, let μ denote additive Haar measure.

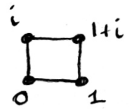
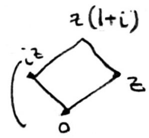
These cho

Unique up to multiplication by a scalar.

Define $|x|_v$ = factor by which x. scales Haar measure.

(e.g. $|x|_v = \dfrac{\mu(xA)}{A}$ for $A \subset K_v$ measurable).

E.g. $K_v = \mathbb{R}$: $\dfrac{\mu(x[0,1])}{\mu([0,1])} = \mu([0,x]) = |x|$.

$K_v = \mathbb{C}$: $B =$  , $z \in \mathbb{C}$: $\dfrac{\mu(zB)}{\mu(B)} = \mu\Big($  $\Big) = |z|^2$.

$K_v = \mathbb{Q}_p$: $\mathbb{Z}_p = \bigsqcup_{0 \le m < p} m + p\mathbb{Z}_p$ $\Rightarrow p\,\mu(p\mathbb{Z}_p) = \mu(\mathbb{Z}_p)$.

$\Rightarrow \dfrac{\mu(p\mathbb{Z}_p)}{\mu(\mathbb{Z}_p)} = \dfrac{1}{p}$!

$\therefore |p| = \frac{1}{p}$.

Whenever we consider the norm on a locally compact field we will always consider the canonical norm.

Notation: $|\cdot|$.

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$\chi : \mathcal{O}_K^\times$ cts ch., $c \in \mathbb{C}$.

Remark: Category of adm...

E.g. $x \mapsto \begin{pmatrix} 1 & \log|x| \\ 0 & 1 \end{pmatrix}$ is a smooth two-dimensional admissible rep which is not semi-simple.

Case of $GL_2(K)$: Recall that we expect roughly two types of representations: "principal series", "cuspidal".

Given cts characters $\chi_1, \chi_2 : K^\times \to \mathbb{C}$ define:

$$I(\chi_1,\chi_2) := \left\{ \varphi : GL_2(K) \to \mathbb{C} \;\middle|\; \begin{array}{l} \varphi \text{ loc. constant} \\ \varphi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot g\right) = \chi_1(a)\chi_2(d)\left|\frac{a}{d}\right|^{1/2}\varphi(g) \\ \forall \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B \end{array} \right\}.$$

Example: $$I(|\cdot|^{-1/2}, |\cdot|^{1/2}) = \left\{ \varphi : GL_2(K) \to \mathbb{C} \;\middle|\; \begin{array}{l} \varphi \text{ loc. constant} \\ \varphi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot g\right) = \varphi(g) \;\forall \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \end{array} \right\}$$

$$= \{\varphi : \underset{\substack{\| \\ B\backslash G}}{\mathbb{P}^1(K)} \to \mathbb{C} \mid \varphi \text{ loc. constant}\}$$

A We saw last time that $I(|\cdot|^{-1/2}, |\cdot|^{1/2})$ is smooth admissible.

$I(\chi_1,\chi_2)$ is called principal series.

Theorem: (a) For all χ_1, χ_2, $I(\chi_1,\chi_2)$ is smooth admissible.

(b) $\widehat{I(\chi_1,\chi_2)} \cong I(\chi_1^{-1}, \chi_2^{-1})$.

(c) If $\chi_1/\chi_2 = |\cdot|^{-1}$ then we have an exact sequence

$$0 \to C(\chi_1,\chi_2) \to I(\chi_1,\chi_2) \to S(\chi_1,\chi_2) \to 0$$

with $\dim C(\chi_1,\chi_2)$ 1-dimensional, $S(\chi_1,\chi_2)$ irreducible.

(d) If $\chi_1/\chi_2 = |\cdot|$ then we have an exact sequence

$$S(\chi_1,\chi_2) \hookrightarrow I(\chi_1,\chi_2) \twoheadrightarrow C(\chi_1,\chi_2)$$

w/ $C(\chi_1,\chi_2)$ one-dimensional, $S(\chi_1,\chi_2)$ irreducible.

(e) Otherwise $I(\chi_1,\chi_2)$ is irreducible.

(f) We have $\underset{\text{if } \chi_1/\chi_2 = |\cdot|^{-1}}{S(\chi_1,\chi_2) \overset{\uparrow}{\cong} S(\chi_2,\chi_1)}$, $\underset{\text{if } \chi_1/\chi_2 = |\cdot|^{1}}{C(\chi_1,\chi_2) \overset{\uparrow}{\cong} C(\chi_2,\chi_1)}$,

and $I(\chi_1,\chi_2) \cong I(\chi_2,\chi_1)$ (if $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$).

Remarks on the theorem:

(a) $I(\chi_1,\chi_2)$ is an example of induction for totally disconnected groups.

(b) Why the strange $\left|\frac{a}{d}\right|^{1/2}$ factor? So that (b) holds!

So why does (b) hold? Consider

$$I(|\cdot|^{\frac{1}{2}}, |\cdot|^{-\frac{1}{2}}) = \left\{ \varphi \mid \varphi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\cdot g\right) = \left|\frac{a}{d}\right| \varphi(g) \right\}.$$

$$\Phi : I(|\cdot|^{\frac{1}{2}}, |\cdot|^{-\frac{1}{2}}) \longrightarrow \mathbb{C}$$

$$\Phi(\varphi) = \int_{K_0} \varphi \, d\mu$$

(One can think of $I(|\cdot|^{\frac{1}{2}}, |\cdot|^{-\frac{1}{2}})$ as being some "functions" on $\mathbb{P}^1(K)$, and we are integrating over $\mathbb{P}^1(K)$ to get a number.)

Minor miracle: Φ is G-invariant, hence $C(|\cdot|^{\frac{1}{2}}, |\cdot|^{-\frac{1}{2}}) = \mathbb{C}$ (trivial rep).

Now given $\varphi \in I(\chi_1,\chi_2)$, $\varphi' \in I(\chi_1^{-1},\chi_2^{-1})$, $\varphi\varphi' \in I(|\cdot|^{\frac{1}{2}}, |\cdot|^{-\frac{1}{2}})$.

$$\Rightarrow I(\chi_1,\chi_2) \times I(\chi_1^{-1},\chi_2^{-1}) \longrightarrow \mathbb{C} \quad \text{G-invariant pairing.}$$

(f) is the most complicated, uses an interhwiner $I(\chi_1,\chi_2) \to I(\chi_2,\chi_1)$ which is defined via analytic continuation (connections to Jantzen filtration).

Finally, note that because Rep $G(K)$ is not semi-simple, we can't just compute Homs as in finite groups case.

6

Cuspidal reps:

Thm: For every degree 2 ext $\begin{matrix} L \\ | \\ K \end{matrix}$ and continuous character $\theta : L^\times \to \mathbb{C}^\times$ (which does not factor through the norm) there exists an irreducible rep. $BC_{L/K}(\theta)$.
We have $BC_{L/K}(\theta) \cong BC_{L/K}(\theta') \Leftrightarrow \theta^\sigma \cong \theta'$ for $\sigma \in Gal(L/K)$.

The construction of $BC_{L/K}(\theta)$ is complicated, via the Weil representation.

What is going on metaphorically?

$SL_2(\mathbb{F}_q)$: $\theta : T_a \to \mathbb{C}^\times \rightsquigarrow \mathcal{L}_\theta$ local system on $\mathbb{P}^1_{\overline{\mathbb{F}}_q} \setminus \mathbb{P}^1(\mathbb{F}_q)$ $\overset{H^1}{\rightsquigarrow} R^G_{T'}(\theta)$.

$SL_2(\mathbb{R})$: $\theta : SO_2 \to \mathbb{C}^\times$ (classified by $n \in \mathbb{Z}$) $\rightsquigarrow$ $\begin{matrix} \mathcal{O}(n) \\ | \\ \mathbb{H} \end{matrix}$ upper half plane $\rightsquigarrow \Gamma(\mathbb{H}, \mathcal{O}(n))$ yields discrete series.

$GL_2(K)$: $\theta : L^\times \to \mathbb{C}^\times \rightsquigarrow$ $\begin{matrix} \mathcal{L}_\theta \\ \downarrow \\ \mathbb{P}^1(\overline{K}) \setminus \mathbb{P}^1(K) \end{matrix}$ ("Drinfeld space") $\rightsquigarrow H^1 = BC_{L/K}(\theta)$.

this theory is still being worked out!

This can also be seen through the lens of Langlands functoriality:

L
$|\deg = n$
K

$W_L \subset W_K$
index n

$$\begin{tikzcd}
\left\{\begin{matrix}\text{1-dim reps} \\ \text{of } L^\times\end{matrix}\right\} \arrow[r, leftrightarrow] \arrow[d, "\operatorname{ind}_{W_L}^{W_K}"] & \left\{\begin{matrix}\text{irred. smooth ad.} \\ \text{reps of } GL_1(L)\end{matrix}\right\} \arrow[d, "BC = \text{"base change"}"] \\
\left\{\begin{matrix}\text{n-dim reps} \\ \text{of } K^\times\end{matrix}\right\} \arrow[r, leftrightarrow] & \left\{\begin{matrix}\text{irred. smooth} \\ \text{reps of } GL_n(K)\end{matrix}\right\}
\end{tikzcd}$$

__Weil-Deligne reps:__ We are almost ready to make a rigorous statement of local Langlands conjecture!

Recall local class field theory: $r_K : W_K \to K^\times$.
Composing w/ $|\cdot|$ gives us the norm character $|\cdot| : W_K \to \mathbb{Q}^\times$.

An __n-dimensional Weil-Deligne representation__ is a triple (ρ, V, N) where

- V is an n-dim. $\mathbb{C}$-v.s.
- $\rho : W_K \to GL(V)$ is a continuous rep;
- $N \in \operatorname{End}(V)$ is nilpotent

such that $\rho(x) N \rho(x)^{-1} \overset{(*)}{=} |x| N$ for all $x \in W_K$.

(In fact $(*)$ forces N to be nilpotent.)

__Examples:__ ① Any n-dim. rep of W_K with $N = 0$.

② $\rho = \begin{pmatrix} |\cdot|\chi & \\ & \chi \end{pmatrix}$, $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

__Note:__ $$\rho(x) N \rho(x^{-1}) = \begin{pmatrix} |x|\chi(x) & 0 \\ 0 & \chi(x) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |x|^{-1}\chi(x)^{-1} & 0 \\ 0 & \chi^{-1}(x) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & |x| \\ 0 & 0 \end{pmatrix} = |x| \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We denote this WD rep: $St(\chi, |\cdot|\chi)$.

A Weil-Deligne representation is <u>F-semi-simple</u> if V is semi-simple as a rep of W_K.

<u>Exercise</u>: Let $\mathrm{Frob} \in W_K$ be any lift of Frobenius.
A WD rep is F-semi-simple $\iff$ Frob is semi-simple.

<u>Theorem</u>: (Local Langlands correspondence for GL_2, $p \neq 2$).
Fix a local field K of residue char. $p \neq 2$. There is a canonical bijection

$$\left\{ \begin{matrix} \text{F-semi-simple} \\ \text{2-dimensional} \\ \text{Weil-Deligne reps} \end{matrix} \right\}_{/\cong} \xleftrightarrow{\sim} \left\{ \begin{matrix} \text{irred. smooth admis.} \\ \text{reps of } GL_2(K) \end{matrix} \right\}_{/\cong}$$

Moreover this bijection is given as follows:

$$\chi_1/\chi_2 = |\,|^{\pm 1}: \left(\begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}, N = 0 \right) \longleftrightarrow C(\chi_1, \chi_2)$$

$$\chi_1/\chi_2 = |\cdot|^{\pm 1}: \left(\begin{pmatrix} \chi|\,| & 0 \\ 0 & \chi \end{pmatrix}, N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \longleftrightarrow S(\chi|\cdot|, \chi)$$

$$\chi_1/\chi_2 \neq |\cdot|^{\pm 1}: \left(\begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}, N = 0 \right) \longleftrightarrow I(\chi_1, \chi_2)$$

L/K
$\theta: L^\times \to \mathbb{C}^\times$
not factoring through norm

$$: \left(\mathrm{ind}_{W_K}^{W_L} \theta, N = 0 \right) \longleftrightarrow BC_{L/K}(\theta).$$