<u>The big picture:</u> <u>Dimension zero</u>

Lets go back to where we started!

$f(x) \in \mathbb{Z}[x]$ :

| p | 2 | 3 | 5 | 7 | 11 | 13 |
| --- | --- | --- | --- | --- | --- | --- |
| # sol$^n$s of $f(x)=0$ mod p | 1 | 0 | 2 | 0 | 0 | 2 |

Via rep theory: $\{\sigma_1, \dots, \sigma_n\}$ roots of $f(x)$ in $\overline{\mathbb{Q}}$.

$\rightsquigarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \curvearrowright H := \bigoplus \mathbb{C}\sigma_i$, for unramified primes

$\#$ sol$^n$s mod p $=$ Trace $(\mathrm{Frob}_p, H)$.

Even in this innocent ("dimension zero") case, $H$ is enormously complicated. Instead we can consider

$\{\sigma_1', \dots, \sigma_n'\}$ roots of $f(x)$ in $\overline{\mathbb{Q}}_p$.

$\rightsquigarrow \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \curvearrowright H_{\mathbb{Q}_p} = \bigoplus \mathbb{C}\sigma_i'$, "local Galois representations".

| p | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| | $H_{\mathbb{Q}_2}$ | $H_{\mathbb{Q}_3}$ | ... | | | | | | |

p unramified $\Leftrightarrow$ inertia acts trivially $\Leftrightarrow$ $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \curvearrowright H_{\mathbb{Q}_p}$ is unramified

$\updownarrow$ satake

Semi-simple conjugacy class $[x] \in GL_n$

$\#$ sol$^n$s mod p $=$ Trace$([x])$.

<u>Slogan:</u> "there's a lot of substance at ramificatied primes / points" (think of maps between complex surfaces).

for ramified primes, $H_{\mathbb{Q}_p}$ can be quite complicated:

1. study of $H_{\mathbb{Q}_p}$ allows us to define local factors in Artin L-function.
2. we can hope to understand $H_{\mathbb{Q}_p}$ via local langlands correspondence.

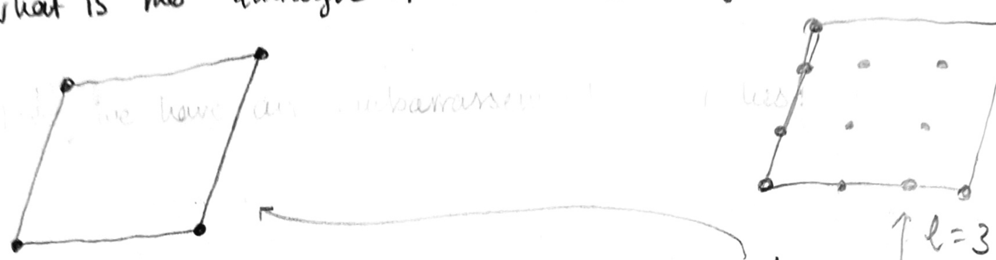
## Dimension $\geq 1$:

As a classic example: consider $E$ (projective completion of) $y^2+y = x^3+x+3x+5$.

(this was the curve we considered when discussing Sato-Tate conjecture.)

$$\begin{array}{c} X \ \text{proper smooth} \\ \downarrow \\ \operatorname{Spec} \mathbb{Z}[\tfrac{1}{N}] \end{array}$$

What is the analogue of $H = \bigoplus \mathbb{C}\sigma_i$ ?




Recall that $E$ is a group, $E(\mathbb{C})$ = solutions over $\mathbb{C}$ $= \mathbb{C}/\Lambda$

For any prime $\ell$ consider $E[\ell^m] = \{x \in E(\overline{\mathbb{Q}}) \mid \ell^m \cdot x = 0\}$

$$\cong (\mathbb{Z}/\ell^m\mathbb{Z})^2 \circlearrowleft \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

Tate module: $\varprojlim E[\ell^m] = T_\ell(E) \cong \mathbb{Z}_\ell^2 \circlearrowleft \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Moreover, if $X_{\mathbb{F}_p}$ is smooth: $\# E(\mathbb{F}_p) = 1 + p - \operatorname{Tr}(\operatorname{Frob}_p, T_\ell(E))$.

$T_\ell(E)$ is an example of "étale cohomology", $T_\ell(E) \cong H^1(E, \mathbb{Z}_\ell)^*$.

$X/k$, $\ell$ prime s.t. $\ell \neq 0$ in $k$, $H^i_{\acute{e}t}(X_{\bar{k}}, \mathbb{Q}_\ell) \circlearrowleft \operatorname{Gal}(k)$

CONTINUOUS!

Again we try to understand $H^i_{\acute{e}t}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ via its

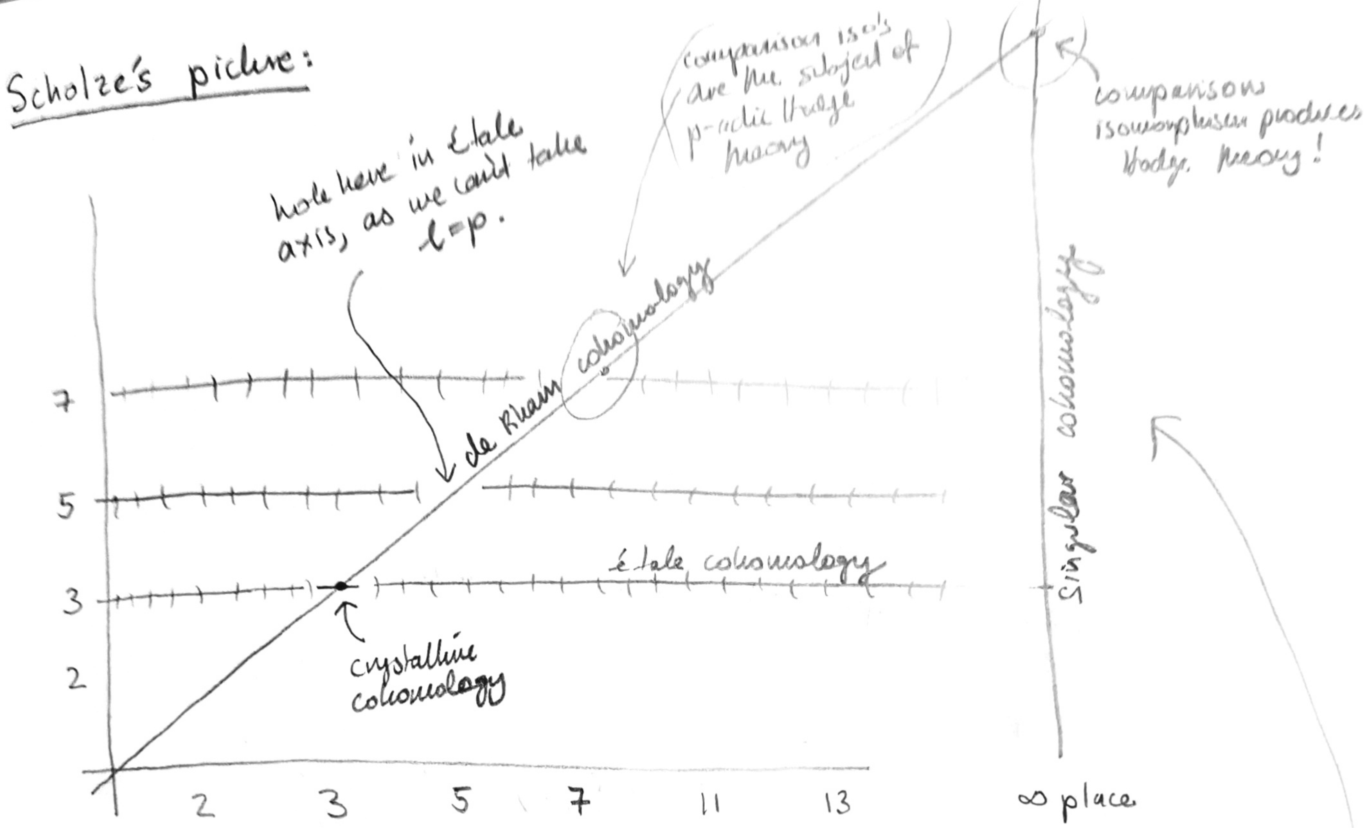
restriction to local Galois groups $\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

We can always calculate these after base change to $\operatorname{Spec} \mathbb{Q}_p$,

and at primes of good reduction after base change to $\operatorname{Spec} \mathbb{F}_p$.

(Exercise: Think about what this statement means for $f(x) \in \mathbb{Z}[x]$.)

Scholze's picture:




One also has other ways of associating cohomology groups to X.

- singular cohomology: $H^i(X(\mathbb{C}), \mathbb{Z})$,
- de Rham cohomology: $H^i_{dR}(X, \mathbb{Z})$, $H^i_{dR}(X_{\mathbb{F}_p}/\mathbb{F}_p)$ cohomology of differential forms.
- crystalline cohomology: $H^i_{crys}(X_{\mathbb{F}_p}/\mathbb{Z}_p)$

Grothendieck's philosophy: these should be incarnations of a unique object, the "motive" of X.

Scholze: perhaps the "motive" is more like a sheaf / local system on Spec $\mathbb{Z} \times$ Spec $\mathbb{Z}$.

Scholze also predicts an archimedean theory for varieties in char. p which has been missing since the beginning of the subject!

How does one compare columns? $\quad G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

$$\rho : G_{\mathbb{Q}_p} \to \mathrm{Gl}_n(\mathbb{Q}_\ell) \quad \text{vs.} \quad \rho' : G_{\mathbb{Q}_p} \to \mathrm{Gl}_n(\mathbb{Q}_{\ell'}).$$

The problem is the topology!

A topological group $\Gamma$ is pro-p if it is profinite

and $\Gamma/N$ is a p-group, for all open normal $N \subseteq \Gamma$.

Examples: (a) Wild inertia $\subset G_{\mathbb{Q}_p}$ $\quad$ (b) $1 + p\,\mathrm{Mat}_n \mathbb{Z}_p = K_1 \subset \mathrm{Gl}_n(\mathbb{Q}_p)$.

Lemma: Any cts $\rho$ : pro-p $\to$ pro-$\ell$ is trivial.

Corollary: Any homomorphism $\rho$ : pro-p $\to \mathrm{Gl}_n(\mathbb{Q}_\ell)$ has finite image.

We have seen $\quad$ : $\quad G_{\mathbb{Q}_p}$
(ramification filtration)

$$\begin{array}{lll} \text{WILD} & ) \ \text{pro-p} & \\ \text{TAME} & ) \ \prod_{\ell' \neq p} \mathbb{Z}_{\ell'} & = \underset{1_\ell \in}{\mathbb{Z}_\ell} \times \prod_{\ell' \neq p,\ell} \mathbb{Z}_{\ell'} \ (\text{pro-p}) \\ \text{UNRAMIFIED} & ) \ \widehat{\mathbb{Z}} & \quad \widehat{\mathbb{Z}} \end{array}$$

Grothendieck: This bit must have finite image!

Moreover, $\rho(1_\ell) \in \mathrm{Gl}_n(\mathbb{Q}_\ell)$ is almost unipotent,

"take logs to get Weil-Deligne rep."

Theorem (Grothendieck) After identifying $\overline{\mathbb{Q}}_\ell = \mathbb{C}$ one has a canonical injection:

$$\left\{ \begin{array}{l} \text{cts. reps.} \\ \rho : G_{\mathbb{Q}_p} \to \mathrm{Gl}_n(\mathbb{Q}_\ell) \end{array} \right\}_{/\cong} \hookrightarrow \left\{ \begin{array}{l} \text{Weil-Deligne reps} \\ \text{of } G_{\mathbb{Q}_p} \text{ over } \mathbb{C} \end{array} \right\}.$$

set is independent of $\ell$.

Rmk: I'm being lazy, one can identify the image.

# Statement of LLC for split groups

$G$ reductive alg. group over $\mathbb{Z}$. $\longleftrightarrow$ root datum $(X^* \supset R,\ X_* \supset R^\vee)$

$\widehat{G}$ dual group $\longleftrightarrow$ ( )

Fix A local field $K$.

A <u>Weil-Deligne rep in $\widehat{G}$</u> is a pair $(\rho, x)$ where

(i) $\rho : W_K \to \widehat{G}(\mathbb{C})$ is cts

(ii) $x \in \mathrm{Lie}\,\widehat{G}$ nilpotent

s.t. $\rho(g) x \rho(g) = |g| x$ for all $g \in W_K$.

A WD rep in $G$ is <u>F-semi-simple</u> if $\rho$ is semi-simple.

<u>Example:</u> A WD rep in $GL_n$ is just an $n$-dim WD rep.

Given $(\rho, x)$ consider $Z_{\widehat{G}}(\rho, x) = \{ g \in \widehat{G}(\mathbb{C}) \mid g \cdot (\rho, x) = (\rho, x) \}$.

<u>LLC</u> There exists a canonical correspondence

$\left\{ \begin{array}{c} \text{F-semi-simple} \\ \text{Weil-Deligne} \\ \text{reps in } \widehat{G} \end{array} \right\} / \widehat{G}\text{-conj.} \xleftrightarrow{1:n} \left\{ \begin{array}{c} \text{irreducible smooth} \\ \text{admissible reps} \\ \text{of } G(K) \end{array} \right\} / \cong \text{iso}$

fibres should be given by irreducible reps of $Z_{\widehat{G}}(\rho, x) / Z_{\widehat{G}}(\rho, x)^\circ$

$\left\{ \begin{array}{c} \text{"tamely ramified w/} \\ \text{unipotent monodromy"} \\ \text{TRUM} \\ \text{i.e. } \rho \text{ factors } W_K \twoheadrightarrow \mathbb{Z}, \\ x \text{ arbitrary} \end{array} \right\} \xleftarrow{1:n} \left\{ \begin{array}{c} \text{reps w/ an} \\ \text{Iwahori fixed} \\ \text{vector} \end{array} \right\}$

$\Vert$

$\{ (s, x) \mid s x s^{-1} = q x \} / \widehat{G}\text{-conjugacy}$

$s$: semi-simple

$x$: nilpotent

$\downarrow s$

reps of $\mathcal{H}(I, G(K))$

(Iwahori-Hecke algebra)

<u>Deligne-Langlands conjecture:</u> Suppose $q \in \mathbb{C}$ is a prime power (or at least $0 \neq q$ is not a root of unity)

$$\left\{ (s,n,\rho) \;\middle|\; \begin{array}{l} s \in \hat{G} \text{ semi-simple} \\ n \in \widehat{\mathrm{Lie}\,\hat{G}} \text{ nilpotent} \\ \rho \in Z_{\hat{G}}(s,n) \end{array} \right\} \Big/ \hat{G}\text{-conjugacy} \xleftrightarrow{\sim} \left\{ \begin{array}{l} \text{irreducible reps of} \\ H_{\mathrm{aff}} \text{ at } q \in \mathbb{C} \end{array} \right\} \Big/ \cong .$$

s.t. $sns^{-1} = qn$

<u>Recall:</u> Unramified LLC followed from Satake isomorphism

$$\mathcal{H}(G(\mathcal{O}_K), G(K)) \xleftrightarrow{\sim} R(\hat{G}) \xleftrightarrow{\sim} K^0(pt/\hat{G})$$

$$R(\hat{G}) = \mathcal{O}(\hat{G}/\hat{G}) = \mathcal{O}\left(\begin{array}{c}\text{semi-simple} \\ \text{conj. classes in } \hat{G}\end{array}\right)$$

"constructible" "coherent"

Tame LLC follows from Kazhdan-Lusztig isomorphism:

$$H_{\mathrm{aff}} := \mathcal{H}(I, G(K)) \xleftrightarrow{\sim} K^{G \times \mathbb{C}^\times}(St)$$

"constructible" "coherent"

<u>Notation:</u>

$$\tilde{\mathcal{N}} = T^*\mathcal{B} \quad \text{Springer resolution}$$
$$\downarrow$$
$$\mathcal{N} \subset \mathrm{Lie}\,\hat{G} \quad \text{nilpotent cone}$$

Steinberg variety

$$\begin{tikzcd} & St \arrow[dl] \arrow[dr] & \\ \tilde{\mathcal{N}} \arrow[dr] & & \tilde{\mathcal{N}} \arrow[dl] \\ & \mathcal{N} & \end{tikzcd}$$

$\mathcal{B} := \hat{G}/\hat{B}$, $\mathcal{B}_e = \pi^{-1}(e)$ Springer fibre.

General nonsense: $K^{\hat{G} \times \mathbb{C}^\times}(St) \curvearrowright K^{Z_{\hat{G}\times\mathbb{C}^\times}(e)}(\mathcal{B}_e)$

$$Z_{\hat{G}\times\mathbb{C}^\times}(e) = \{(g,c) \in \hat{G} \times \mathbb{C}^\times \mid c \cdot geg^{-1} = e\}, \quad c \cdot geg^{-1} = e \iff geg^{-1} = c^{-1}e$$

$\rightsquigarrow$ thus K-theory of Springer fibres provides all simple $H_{\mathrm{aff}}$-modules, proving Deligne-Langlands conjecture.

## Geometric Satake equivalence:

From now on $K = k((t)) \supset \mathcal{O}_K = k[[t]]$.

$$\mathcal{H}(G(\mathcal{O}_K), G(K)) = G(\mathcal{O}_K)\text{-invariant functions on } G(K)/G(\mathcal{O}_K)$$

$\uparrow$ $k$-points of an ind-variety "affine Grassmannian". $Gr_G$

$k = \mathbb{F}_q$ or $\mathbb{C}$ ...

$$(\mathrm{Perv}_{G(\mathcal{O}_K)}(Gr_G; \mathbb{C}), *) \cong (\mathrm{Rep}\, {}^L G_{\mathbb{C}}, \otimes).$$

"constructible" "coherent"

→ major tool in geometric rep. theory → key to recent work of V. Lafforgue giving global "automorphic to Galois" correspondence for function fields.

## Bezrukavnikov's equivalence:

$$\mathcal{H}(I, G(K)) = I\text{-invariant functions on } G(K)/I$$

$\uparrow$ $k$ points of affine flag variety. $\mathcal{F}l_G$.

## Bez's equivalence:

convolution (different) convolution.

$$(D^b_I(\mathcal{F}l_G; \mathbb{C}), *) \xrightarrow{\sim} (D^b\, \mathrm{Coh}^{G \times \mathbb{C}^*}(\mathrm{St}), *)$$

"constructible" "coherent".

**Remarks:** I'm lying ... it will take some more lectures to explain exactly what the correct categories are on both sides!