FRIDAY 1

$G/\mathbb{Z}$ Chevalley scheme
$k = \bar{k}$ field of char. p

$Gr = G^\vee((t))/K$
affine Grassmannian

(2017)
Achar–Riche + many others

(Lusztig conjecture for G_k in char p (1980)) $\Longleftrightarrow$ (absence of p-torsion in $IC(\overline{Gr_x}, \mathbb{Z})$ for $\{x \mid x^{-1}_p : 0 \leq 2\text{-}p\text{-adic digits}\}$)

$\Updownarrow$ compatibility w/ Kato (1985) S∞ Thm.

(" ——— $\{x \mid x^{-1}_p : 0 \leq 1$ p-adic digit$\}$)

"independence of p" $\Downarrow$

Lusztig conjecture for large p
(absence of torsion in finitely-many cohomology groups)

NOT effective!

First proof KT, KL, L, AJS (1995).

Fiebig (late 2000s): effective
enormous bound independent of p.

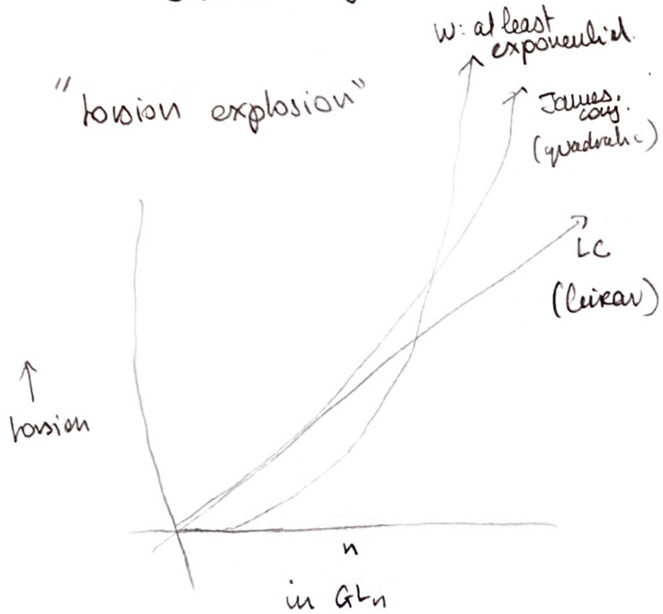
$\Downarrow$ ①

(absence of p-torsion in $IC(X_x, \mathbb{Z})$ for all $X_x \subset G^\vee/B^\vee$ in $p > h$
finite flags)

$\Downarrow$ W + friends (2013)

counter-examples to expected bounds in Lusztig conjecture + James conjecture for GL_n.

"torsion explosion"



① Follows from "Steinberg embedding":

For any $\lambda \in X_{++}$,

$G^\vee/B^\vee \hookrightarrow Gr$

$g \longmapsto g \cdot t^\lambda$

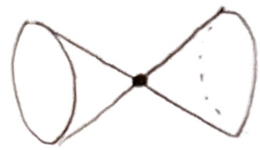
(stratum preserving). $\quad Gr \supset$

$\mathrm{Perv}(G^\vee/B^\vee) \simeq \mathrm{Perv}\left(\bigsqcup_{x \in W_f} Iw \cdot t^{x\lambda}\right)$

(locally closed).

$\boxed{\text{FRIDAY 2}}$

$$\mathbb{C}^2/(\pm 1)$$
$$\|$$
$$\operatorname{Spec} \mathbb{C}[X,Y]^{(\pm 1)}$$
$$\|$$
$$\operatorname{Spec} \mathbb{C}[\underset{a}{\underset{\|}{X^2}}, \underset{c}{\underset{\|}{XY}}, \underset{b}{\underset{\|}{Z^2}}] = \operatorname{Spec} \mathbb{C}[a,b,c]/(ab-c^2)$$



$$= \left\{ x = \begin{pmatrix} c & -a \\ b & -c \end{pmatrix} \subset \mathfrak{sl}_2(\mathbb{C}) \;\middle|\; x \text{ is nilpotent} \right\}$$

Exercises: sits inside the affine Grassmannian for SL_2.

$X = X^{reg} \sqcup \{0\}$ (stratification)

$\mathcal{F} \in \operatorname{Perv}_S(X,k)$

$H^i(\mathcal{F}|_{X_s})$

	-3	-2	-1	0	1
X_{reg}	0	*	0	0	0
$\{0\}$	0	*	*	*	0

degrees allowed for a perverse sheaf.

Computing $IC(\overline{X^{reg}}, k)$ Use Deligne construction:

$$\tau_{<0}\, j_* \underline{k}_{X_{reg}}[2].$$

First compute $j_* \underline{k}_{X_{reg}}[2]$.

$$S^3_\varepsilon \cap X_{reg} \cong S^3/\pm 1 = \mathbb{RP}^3.$$

$$\left(j_* \underline{k}_{X_{reg}}[2]\right)_0 = \lim_{\varepsilon \to 0} H^{i+2}(B(0,\varepsilon) \cap X_{reg}, k)$$

$$H^*(\mathbb{RP}^3, \mathbb{Z}) = \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \hline \mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z} \end{array} \rightsquigarrow H^*(\mathbb{RP}^3, k) = \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \hline k & (k)_2 & (k)_2 & k \end{array}$$

$(k)_2$: 0 if $2 \neq 0$ in k, k otherwise.

Stalks of $j_* \underline{k}_{X_{reg}}[2]$ are

-2	-1	0	1
k			
k	$(k)_2$	$(k)_2$	k

FRIDAY-3

$IC(\overline{X}_{reg}, k)$

k	0	0
k	$k_{(c)}$	0

Rmk: $IC(\overline{X}, k) = \underline{k}[2]$ can also be seen because X has $\mathbb{Z}/2\mathbb{Z}$ quotient singularities.

Similarly:

$\mathbb{D}$

$IC(X, \mathbb{Z})$ $= \underline{\mathbb{Z}}_X[2]$

-2	-1	0
$\mathbb{Z}$	0	0
$\mathbb{Z}$	0	0

$IC^+(X, \mathbb{Z})$

$\mathbb{Z}$	0	0
$\mathbb{Z}$	0	$\mathbb{Z}/2$

causing all the trickiness!

$IC(X, \mathbb{Z}) \overset{L}{\underset{\mathbb{Z}}{\otimes}} k$ $\to$ simple if char $k \neq 2$

$\to \dfrac{IC(X; k)}{IC(0)}$ if char $k = 2$.

Under geometric Satake:

$\nabla_{2\varpi_1}$ is irreducible $\Leftrightarrow p \neq 2$

Under Finkelberg-Mirkovic:

$2p$ has ≥ 2 p-adic digits $\Leftrightarrow$ (LCF) holds for L_{2p} for Sl_2.

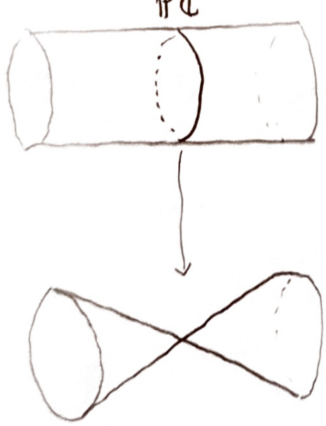
$\Updownarrow$

$p = 2$.

Intersection forms:

$\boxed{\text{Problem is that it is basically impossible to perform calculations w/ the Deligne construction in all but the simplest situations. Instead we need to work w/ resolutions.}}$

$\tilde{X} =$



$\leftarrow$ total space of $\mathcal{O}(-2)$ on $\mathbb{P}^1 = T^*\mathbb{P}^1$.

Crucial observation: $f_* \underline{k}_{\tilde{X}}[2]$ is semi-simple

$\Updownarrow$

$[\mathbb{P}^1]^2$ is invertible in k

$= (-2)$. $\leftarrow$ "intersection form"

In general
$$\begin{tikzcd} \tilde{X} \arrow[d, "f"] & \tilde{N}_\lambda \arrow[l] \arrow[d] & F_\lambda \arrow[l, leftrightarrow] \arrow[d] \\ X = \bigsqcup X_\lambda & N_\lambda \arrow[l, phantom, "\supset"] & x_\lambda \arrow[l, phantom, "\ni"] \end{tikzcd}$$

Assume f semi-small, then $\dim F_\lambda \leq \frac{1}{2} \dim \tilde{N}_\lambda = n_\lambda$

$\rightsquigarrow$ intersection form on $\underset{\underset{2 d n_\lambda}{\shortparallel}}{H_{top}}(F_\lambda) = \underset{\substack{\text{irred. comp.} \\ \text{of } \dim = \frac{1}{2}\dim \tilde{N}_\lambda}}{\bigoplus} \mathbb{Z}[C]$

$f_* \underline{k}[\dim_{\mathbb{C}} \tilde{X}]$ is semi-simple $\iff$ all IF's are non-degenerate over k + local systems are irreducible.

Problem: Intersection forms are usually still very difficult to calculate. (Finding fibre tricky, finding components tricky, not smooth, etc.)

Miracle situation: F_λ is irreducible smooth. $IF_\lambda = ~~~~$
$= p_!$ (Euler class of normal bundle of $F_\lambda \subset \tilde{N}_\lambda$)

Torsion explosion (W) For any entry γ of any word of length ℓ in the generators $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ one can associate a Schubert variety $X_x \subset SL_{3\ell+5}/B$, and a (Bott-Samelson) resolution $\tilde{X}_x \to X$, and a point $w_I \in X$ s.t. the miracle situation holds, and $IF = (\pm\gamma)$.

Some number theory: torsion in stalks and costalks of Schubert varieties in SL_n/B grows at least exponentially in n.