Part 12 29/11/13

Soergel 'On the relation between intersection cohomology and representation theory in positive characteristic' explains how Soergel bimodules with positive-characteristic coefficients control part of the Lusztig conjecture. We will now see how intersection forms play a role in positive characteristic also, in particular in calculating the p-canonical basis.

Let $(X, \Phi, X^*, \Phi^\vee)$ be a root datum, $\Delta \subset \Phi$, $\Delta^\vee \subset \Phi^\vee$ a set of simple roots & corresponding set of simple coroots. Let $W \supset S$ be the corresponding Weyl group and set of simple reflections.

Then $\underline{h} = X^*$, $\Delta \subset \underline{h}^*$, $\Delta^\vee \subset \underline{h}$ is a realization of (W,S) over $\underline{\mathbb{Z}}$.

(More generally, we could start with any generalized Cartan matrix.)

Form the resulting diagrammatic category $\mathcal{D}$, defined over $\mathbb{Z}$. So given any commutative ring $\Bbbk$, we can obtain $\mathcal{D}_\Bbbk$ by base change (i.e. $\mathrm{Hom}_{\mathcal{D}_\Bbbk}(x,y) = \mathrm{Hom}_{\mathcal{D}}(x,y) \otimes_{\mathbb{Z}} \Bbbk$).

Let $\hat{\mathcal{D}}_\Bbbk = (\mathcal{D}_\Bbbk)_{\oplus, (m), \mathrm{Kar}}$.

Soergel categorification theorem (Assume $\Bbbk$ is a complete local ring.): $\mathrm{ch}: [\hat{\mathcal{D}}_\Bbbk] \xrightarrow{\sim} \mathcal{H}$

$\{\text{Indecomposable objects in } \hat{\mathcal{D}}_\Bbbk\}/\text{shifts, isom} \longleftrightarrow W$

$B_w \longleftrightarrow w$

${}^p\underline{H}_w = \mathrm{ch}(B_w)$ only depends on the residue characteristic p of $\Bbbk$. This is the 'p-canonical basis' of $\mathcal{H}$.

Note: For $p=2$, B_3 and C_3 give rise to different p-canonical bases, so this does depend on the realization of (W,S) not just on the underlying Coxeter system.

<u>Properties:</u> 1) ${}^p\underline{H}_x = \underline{H}_x$ for $p \gg 0$.

2) ${}^p\underline{H}_x \, {}^p\underline{H}_y = \sum_z {}^p\mu_{xy}^z \, {}^p\underline{H}_z$ for ${}^p\mu_{xy}^z \in \mathbb{N}[v^{\pm 1}]$ (not unimodal in general)

3) ${}^p\underline{H}_x = \sum_y {}^p m_{yx} \underline{H}_y$ for ${}^p m_{yx} \in \mathbb{N}[v^{\pm 1}]$

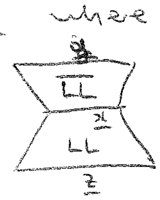
Motivation: X variety, $Z \subset X$ closed subset, $U = X \setminus Z$

For various concepts of sheaves, $Sh(U) = Sh(X)/Sh(Z)$.

To decompose Bott-Samelson $B_{\underline{w}}$, which corresponds to a parity sheaf on the Schubert variety labelled by w, we use this idea: focus on an open set of all things $\geq x$.

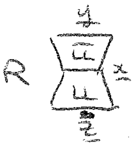
Let $I \subset W$ be a lower order ideal for Bruhat order.

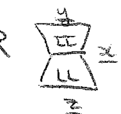
Let $\mathcal{D}_I$ be the ideal of $\mathcal{D}$ generated by all morphisms factoring through $\underline{x}$ where $\underline{x}$ is a rex for $x \in I$.



For $J \subset W$ an upper order ~~[illegible]~~ ideal,

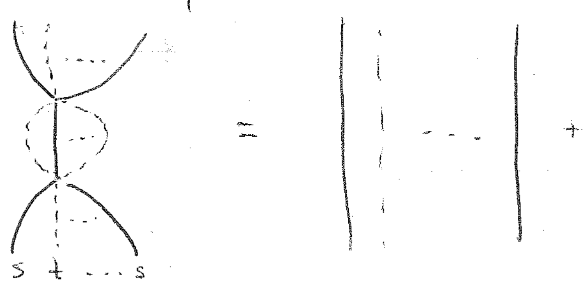
let $\mathcal{D}^J = \mathcal{D}/\mathcal{D}_{W \setminus J}$.

The double leaves theorem $\Rightarrow$ $\mathrm{Hom}_{\mathcal{D}_I}(\underline{z}, \underline{y}) = \bigoplus_{x \in I} R$ 

$\mathrm{Hom}_{\mathcal{D}^J}(\underline{z}, \underline{y}) = \bigoplus_{x \in J} R$ 

The most important case is where J is principal, i.e. $J = \{y \in W \mid y \geq x\}$, in which case we write $\mathcal{D}^{\geq x}$.

Because all loops in the graph of reduced expressions for x are generated by Zamolodchikov cycles, and



terms belonging to $\mathcal{D}_{< st \cdots s}$,

any two braid moves $\underline{x} \overset{\beta}{\underset{\beta'}{\rightrightarrows}} \underline{x}'$ between reduced expressions for x become equal in $\mathcal{D}^{\geq x}$. Hence we get a canonical object $x \in \mathcal{D}^{\geq x}$, independent of choice of rex.

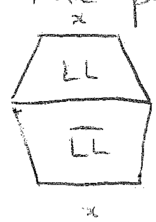
<u>Intersection forms:</u> Fix x, any expression $\underline{w}$, and $d \in \mathbb{Z}$.

$$I_{\underline{w},x,d} : \mathrm{Hom}_{\mathcal{D}^{\geq x}}(\underline{x}(d), \underline{w}) \times \mathrm{Hom}_{\mathcal{D}^{\geq x}}(\underline{w}, \underline{x}(d)) \longrightarrow \mathrm{End}_{\mathcal{D}^{\geq x}}(\underline{x}(d)) = \mathbb{Z}$$

Has a basis $\{\overline{LL}_{\underline{w},\underline{e}} \mid \underline{e} \in M(\underline{w}, x, d)\}$ Has a basis $\{LL_{\underline{w},\underline{e}} \mid \underline{e} \in M(\underline{w}, x, d)\}$

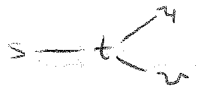
where $M(\underline{w}, x, d) = \{\text{subexpressions } \underline{e} \text{ of } \underline{w} \mid \underline{w}^{\underline{e}} = x,\ \mathrm{def}(\underline{e}) = d\}$

To compute the pairing, stack the diagrams



, use relations to simplify, discard anything factoring through something not $\geq x$.

e.g. for W of type D_4



, $\underline{w} = suvtsuv$, $x = suv$, $d = 0$,

$M(\underline{w}, x, 0)$ has three elements, and the form has matrix $\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$, with invariant factors 1,1,2

correspondingly, when $p=2$, ${}^p\underline{H}_{suvtsuv} = \underline{H}_{suvtsuv} + \underline{H}_{suv}$ (Braden)

<u>Lemma</u> Multiplicity of $B_x(d)$ in $B_{\underline{w}}$ is the rank of $I_{\underline{w},x,d} \otimes_{\mathbb{Z}} (\text{residue field of } \Bbbk)$.

<u>Algorithm:</u> Fix a rex $\underline{x}$ for all $x \in W$.

Use the lemma to write $[B_{\underline{w}}] = \sum_x m_{x,\underline{w}} [B_x]$

with $m_{x,\underline{w}} = 1$ if $x = w$

$m_{\underline{x},\underline{w}} = 0$ unless $x \leq w$.

Because $\mathrm{ch}(B_{\underline{w}}) = \underline{H}_{\underline{w}}$, conclude that

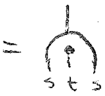
$$\underline{H}_{\underline{w}} = \sum_x m_{x,\underline{w}}\, {}^p\underline{H}_x .$$

Then invert the matrix to calculate ${}^p\underline{H}_x$.

<u>Example</u> $\underline{w} = sts$. The only nonempty forms are:

$I_{\underline{w},w,0} = (1)$, $I_{\underline{w},s,0} = (\langle \alpha_s, \alpha_t^\vee \rangle)$

$M(\underline{w}, s, 0) = \{100\}$, $LL_{\underline{w},100} =$



So $\underline{H}_{\underline{w}} = \begin{cases} {}^p\underline{H}_w + {}^p\underline{H}_s & \text{if } \langle \alpha_s, \alpha_t^\vee \rangle \neq 0 \mod p \\ {}^p\underline{H}_w & \text{if } \langle \alpha_s, \alpha_t^\vee \rangle = 0 \mod p \end{cases}$