Part 13 3/12/13

Intersection forms and the nil Hecke ring

Let $Q = \text{Quot } R$, consider $X_s = \frac{1}{\alpha_s}(1-s) \in Q \rtimes W$.

<u>Lemma</u> 1) X_s's satisfy the braid relations

2) $X_s^2 = 0$.

So we get well-defined elements X_w for any $w \in W$, and

$$X_{x_1} X_{x_2} = \begin{cases} X_{x_1 x_2} & \text{if } l(x_1)+l(x_2) = l(x_1 x_2) \\ 0 & \text{otherwise.} \end{cases}$$

Now Q is a $Q \rtimes W$ module via $(wq)q' = w(qq')$.

Define the nil-Hecke ring $NH = \{ a \in Q \rtimes W \mid a(R) = R \}$.

Note that for $f \in R$, $X_s(f) = \partial_s(f)$, so $X_s \in NH$.

<u>Thm</u> (Kostant-Kumar) NH is free as a left (or right) R-module with basis $\{X_x \mid x \in W\}$.

<u>Remark</u> (Also Kostant-Kumar) when the realization is crystallographic (as here), $NH^{*} = H_T^*(G/B)$, $NH^* \otimes_R k = H^*(G/B)$.
(dual)

i.e. NH^* is the correct generalization of the coinvariant ring.

Alternatively, NH has a presentation

$$\langle r \in R, X_s, s \in S \mid r X_s = X_s (sr) + \partial_s(r) \rangle$$

and is graded via $\deg h^* = 2$, $\deg X_s = -2$.

There are special situations where the light-leaves maps are independent of choice: namely, where there are no D1's in the subexpression.

Take $\underline{w} = s_1 s_2 \cdots s_m$, two subexpressions $e_1 \cdots e_m, e_1', \cdots, e_m'$ for the same x, with no D1's, $\text{def}(\underline{e}) + \text{def}(\underline{e}') = 0$.

Define an element of NH by

$$\delta\binom{e_i}{e_i'} = \begin{cases} \alpha_{s_i} & \text{if } e_i, e_i' \text{ are both U0} \\ 1 & \text{if one of } e_i, e_i' \text{ is U0} \\ X_{s_i} & \text{otherwise} \end{cases},$$

let $\Delta = \delta\binom{e_1}{e_1'} \cdots \delta\binom{e_m}{e_m'}$.

<u>Example</u> $\underline{w} = sts$, $x = s$, $d = 0$, $\underline{e} = \underline{e}' = (1,0,0)$.
U1 U0 D0

$$\Delta = X_s \alpha_t X_s = X_s \partial_s(\alpha_t) = X_s \langle \alpha_t, \alpha_s^\vee \rangle.$$

<u>Exercise</u> $\Delta = (\text{scalar}) X_x + \sum_{y<x} c_y X_y$ for some $c_y \in R$, and $\deg(\Delta) = -2l(x)$,

so $\Delta = (\text{scalar}) X_x$.

Thm (He – W.) Under the above assumptions on $\underline{e}, \underline{e}'$,

$$\Delta = I_{\underline{w},x,d}(LL_{\underline{e}}, LL_{\underline{e}'})\, X_x \quad \text{in } NH.$$

Example In A_7, consider

$\underline{w} = 1\,3\,2\,4\,3\,5\,4\,3\,2\,1\,6\,7\,6\,5\,4\,3$

$x = w_I$ where $I = \{1,3,4,5,7\}$

Only one defect-0 subexpression:

$\underline{e} =$ U1 U1 U0 U1 U1 U1 U1 U1 U0 D0 U0 U1 U0 D0 D0 D0

To abbreviate, write 1 for X_1, 2 for X_2, etc., circle constants.

$$\Delta = 1\,3\,\alpha_2\,4\,3\,5\,4\,3\,\alpha_2\,1\,\alpha_6\,7\,\alpha_6\,5\,4\,3 \qquad [\alpha_6 7 = 7(\alpha_6+\alpha_7) - 1]$$

$$= -1\,3\,\alpha_2\,4\,3\,5\,4\,3\,\alpha_2\,1\,\alpha_6\,5\,4\,3$$
$$+ 1\,3\,\alpha_2\,4\,3\,5\,4\,3\,\alpha_2\,1\,7\,\alpha_6^2\,5\,4\,3$$
$$+ 1\,3\,\alpha_2\,4\,3\,5\,4\,3\,\alpha_2\,1\,7\,\alpha_6\,5\,4\,3\,\alpha_7$$

$$= 1\,3\,\alpha_2\,4\,3\,5\,4\,3\,\alpha_2\,1\,7\,\alpha_6^2\,5\,4\,3$$ since other terms must vanish by the Exercise. $[\alpha_2 1 = 1(\alpha_1+\alpha_2) - 1]$

$$= -1\,3\,\alpha_2\,4\,3\,5\,4\,3\,7\,\alpha_6^2\,5\,4\,3$$
$$+ 1\,3\,\alpha_2\,4\,3\,5\,4\,3\,1\,7\,\alpha_6^2\,5\,4\,\alpha_2\,3$$

$\vdots$

$$= 2\;1\,4\,3\,4\,5\,4\,3\,7.$$

similar to Braden's example.

For simplicity, assume $W = S_n$, $R = \mathbb{Z}[y_1, \dots, y_n]$.
NH acts on R and preserves (R_+^W), hence it acts on $H = R/(R_+^W)$
$= H^*(GL_n(\mathbb{C})/B, \mathbb{Z})$.

- Let $Y_{w_0} = y_1^{n-1} \cdots y_{n-1}$, $Y_x = \partial_{xw_0} Y_{w_0}$ "Schubert polynomials".
  Then $H = \bigoplus_{x \in S_n} \mathbb{Z} Y_x$, and $\partial_i Y_x = \begin{cases} Y_{s_i x} & \text{if } s_i x < x \\ 0 & \text{if } s_i x > x \end{cases}$.

For $f \in R$ of degree 2, the Chevalley formula says

$$f \cdot Y_x = \sum_{\substack{t \in T \\ \ell(tx) = \ell(x)+1}} \langle f, \alpha_t^\vee \rangle Y_{tx}.$$

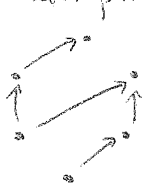
set of reflns in W

Now $X_x \cdot Y_{x^{-1}} = 1$, so in the context of the above Theorem,

$$\Delta \cdot Y_{x^{-1}} = I_{\underline{w},x,d}(LL_{\underline{e}}, LL_{\underline{e}'}).$$

Consider what happens if you start with a Y_x and only allow multiplication by y_1 and y_n, and ∂_i's.

$n=2$: only $\pm Y_x$ occur.

$n=3$: mult. by Y_1 for instance is  in the Bruhat graph —

still only get coefficients of ± 1.

In $n=4$, the story is different:

consider $F = \partial_1 \partial_3 y_1(-y_4)\, \partial_1 \partial_2\, y_1(-y_4) \longrightarrow$.

Then of the length-2 elements,

$$F(Y_{13}) = F(Y_{23}) = F(Y_{32}) = 0,$$

F has matrix $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ on $\mathbb{Z}Y_{12} \oplus \mathbb{Z}Y_{21}$.

matrix governing the Fibonacci recursion, so Fibonacci numbers occur as coefficients.

In $n \geq 5$, one can find large prime factors of coefficients, hence in entries of intersection forms for (highly non-reduced) expressions in S_n.

By increasing the rank from S_n to S_{n+b} one can isolate these entries in a 1×1 intersection form, hence they give torsion primes.

For simplicity, just consider multiplication by y_n: suppose

$X_{w_1} y_n X_{w_2} y_n \cdots y_n X_{w_{b+1}} = C X_x$, $\quad w_i \in S_n$.

Let b be the number of occurrences of y_n, and extend S_n to S_{n+b}.

We can then replace y_n by $\alpha_n = y_n - y_{n+1}$.

C occurs as an entry in an intersection form for

$\underline{w} = w_1 s_n w_2 s_n w_3 \cdots s_n w_{b+1}$, x as above.

Consider $\underline{w}' = w_1 s_n w_2 (s_{n+1} s_n) w_3 \cdots (s_{n+b-1} \cdots s_n) w_{b+1}$.

Claim 1: $\underline{w}'$ is reduced.

Claim 2: there is a unique defect-0 subexpression of $\underline{w}'$ for $x w_J$ where $J = \{n+1, \cdots, n+b-1\}$ and it has no D1's.

Claim 3: $I_{\underline{w}', x w_J, 0}(LL_{\underline{e}}, LL_{\underline{e}}) = \pm C$.

(use Thm again)