Part 3 11/10/13

String diagrams

$\mathcal{A}, \mathcal{B}$ categories, $\mathcal{A} \underset{F}{\overset{\hat{F}}{\rightleftarrows}} \mathcal{B}$ an adjunction.

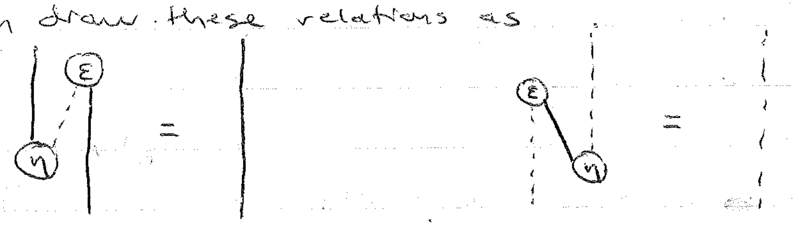
Let $\varepsilon \in \mathrm{Hom}(\hat{F}F, \mathrm{id}_{\mathcal{B}})$ be the counit and
$\eta \in \mathrm{Hom}(\mathrm{id}_{\mathcal{A}}, F\hat{F})$ be the unit of the adjunction.

The compositions

$$F \xrightarrow{\eta F} F\hat{F}F \xrightarrow{F\varepsilon} F \qquad \text{and} \qquad \hat{F} \xrightarrow{\hat{F}\eta} \hat{F}F\hat{F} \xrightarrow{\varepsilon\hat{F}} \hat{F}$$

are the identity ("zig-zag relations")

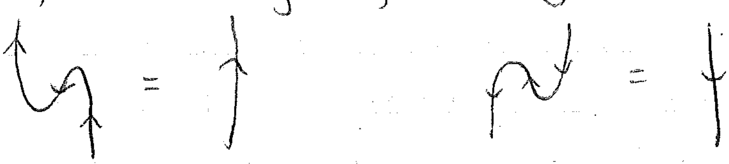
We can draw these relations as



where | denotes id_F and ⁝ $\mathrm{id}_{\hat{F}}$. Here the horizontal slices represent compositions of functors and the vertical stacking represents compositions of natural transformations.

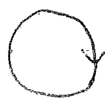
More visually intuitive is to write id_F as ↓ and $\mathrm{id}_{\hat{F}}$ as ↑. Then the relations become the isotopies

If $(F, \hat{F})$ are biadjoint, i.e. $F \dashv \hat{F}$ also, then we also get



Biadjointness, together with the fact that Cat satisfies the axioms of a 2-category, implies that any isotopy class of planar oriented non-intersecting lines determines a well-defined natural transformation. More generally one can define a biadjunction (namely, a pair of 2-morphisms satisfying the zig-zag relations) between 1-morphisms going in opposite directions in any 2-category, and then any isotopy class of planar oriented non-intersecting lines determines a well-defined 2-morphism.

For example:

 represents $\mathrm{id} \xleftarrow{\varepsilon} \hat{F}F \xleftarrow{\eta'} \mathrm{id}$ ← unit of other adjunction

Frobenius extensions

Now suppose specifically that $\mathcal{A} = A\text{-mod}$, $\mathcal{B} = B\text{-mod}$ where $A \subset B$ is an inclusion of rings, $F = \mathrm{res}$ and $\hat{F} = \mathrm{ind} = B \otimes_A -$.

Basic fact: right-continuous functors on module categories form a (right-exact & commuting w/ direct limits) 2-category equivalent to bimodules (objects: rings, 1-morphisms: bimodules, 2-morphisms: maps of bimodules). (Caveat: functors form a strict 2-category and bimodules not.)

i.e. Any right-cts $F: A\text{-mod} \to B\text{-mod}$ is isomorphic to $B_F \otimes -$ for some (B,A)-bimodule B_F, and we have $\mathrm{Hom}_{funct}(F, G) \cong \mathrm{Hom}_{Bim}(B_F, B_G)$.

e.g. In our situation of an inclusion $A \subset B$,

res $\longleftrightarrow$ bimodule ${}_A B_B$ and ind $\longleftrightarrow {}_B B_A$.

The counit is then the multiplication map ${}_B B \otimes_A B_B \to {}_B B_B$ and the unit is the inclusion ${}_A A_A \to {}_A B_A$.

Def $A \subset B$ is a Frobenius extension if we have a fixed adjunction in the other direction res ⊣ ind, i.e. we want maps

$$\partial : B \to A \text{ of } (A,A) \text{ bimodules}$$ (something like a trace)

and $\Delta : B \to B \otimes_A B$ of (B,B) bimodules
(i.e. an element $\Delta(1) \in B \otimes_A B$ s.t. $b\Delta(1) = \Delta(1)b$ for $b \in B$)

satisfying the zig-zag relations, i.e. $(\mathrm{id} \otimes \partial)(\Delta(1)) = 1$, $(\partial \otimes \mathrm{id})(\Delta(1)) = 1$

The isom $\mathrm{Hom}_A(\mathrm{res}(M), N) \cong \mathrm{Hom}_B(M, \mathrm{ind}(N))$ is then given by $(\partial \otimes \mathrm{id})\circ\varphi \longleftarrow\!\!| \varphi$.

Note that this implies that $\mathrm{Hom}_A(B, -)$ $\cong B \otimes_A -$ is right exact, i.e. B is projective as an A-module, and also that $\mathrm{Hom}_A(B,-)$ commutes with injective direct limits, i.e. B is fin. gen. over A.

In fact, ∂ determines Δ (Rouquier, '2-Kac-Moody algebras', §2.3):

Claim: if B is a f.g. projective A-mod. and $\partial : B \to A$ is a map of (A,A)-bimodules such that $(b, b') \mapsto \partial(b'b)$ is nondegenerate in the sense that

$$b \mapsto (b' \mapsto \partial(b'b)) : B \to \mathrm{Hom}_A(B, A) \text{ is an isomorphism,}$$

then ∂ is the counit of a unique Frobenius extension.

If B is free over A with basis $e_1, \dots, e_m$ and $e_1^*, \dots, e_m^*$ is the dual basis with respect to $(b, b') \mapsto \partial(b'b)$ (i.e. $\partial(e_i e_j^*) = \delta_{ij}$), then we must put

$$\Delta(1) = \sum e_i^* \otimes e_i, \text{ the "Casimir".}$$

Exercise: show that $b\Delta(1) = \Delta(1)b$ for all $b \in B$;
$(\mathrm{id} \otimes \partial)(\Delta(1)) = 1$, $(\partial \otimes \mathrm{id})(\Delta(1)) = 1$.

Back to Soergel bimodules and the relations in the rank-1 case.

To study Soergel bimodules, we only need the sub-2-category of bimodules where the rings are the invariant rings $R^I = R^{W_I}$ for $I \subseteq S$, and the bimodule itself is always R, notation as before. The relevance of the above theory comes from the fact that when $I \subseteq J \subseteq S$,

$R^J \subset R^I$ is a Frobenius extension.

Simplest case:

<u>Prop</u> For $s \in S$, $R^s \subset R$ is a Frobenius extension with counit

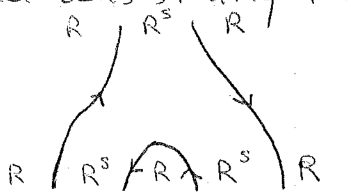
$$\partial : R \longrightarrow R^s : f \longmapsto \frac{f - s(f)}{\alpha_s} \qquad (\text{"Demazure operator"}).$$

<u>Proof</u>: R is a free rank-2 R^s-module with basis $1, \alpha_s$.

A dual basis is $\frac{1}{2}\alpha_s, \frac{1}{2}1$, since $\partial(1) = 0$, $\partial(\alpha_s) = 2$, $\partial(\alpha_s^2) = 0$. □

Note that $\Delta(1) = \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$, the element that arose before.

Therefore we can unambiguously represent certain maps between (R,R)-bimodules or (R,R^s) or (R^s,R) or (R^s,R^s) defined using tensor products $\otimes_R$ and $\otimes_{R^s}$ of factors R, by a planar diagram with the appropriately labelled regions at top & bottom and consistently throughout, e.g.

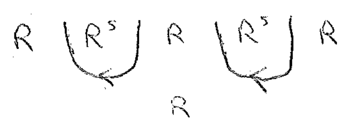


represents the map

$R \otimes_{R^s} R$ ← considered as (R,R)-bimodule

$\uparrow \mathrm{id} \otimes \partial \otimes \mathrm{id}$

$R \otimes_{R^s} R \otimes_{R^s} R$



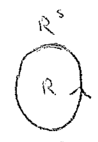
represents the map

$\uparrow (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = \frac{1}{4}(\alpha_s \otimes \alpha_s \otimes 1 + 1 \otimes \alpha_s^2 \otimes 1 + \alpha_s \otimes 1 \otimes \alpha_s + 1 \otimes \alpha_s \otimes \alpha_s)$

$R \qquad 1$

and their composition is $R \longrightarrow R \otimes_{R^s} R : 1 \longmapsto \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$.

Note that inside such diagrams,



represents

R^s ← i.e. unit for $\otimes_{R^s}$

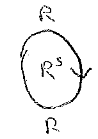
$\uparrow \partial$ — i.e. $R \otimes_R R$ considered as (R^s, R^s)-bim.

R i.e. zero

$\uparrow$ inclusion

R^s

and



represents

R

$\uparrow$ mult

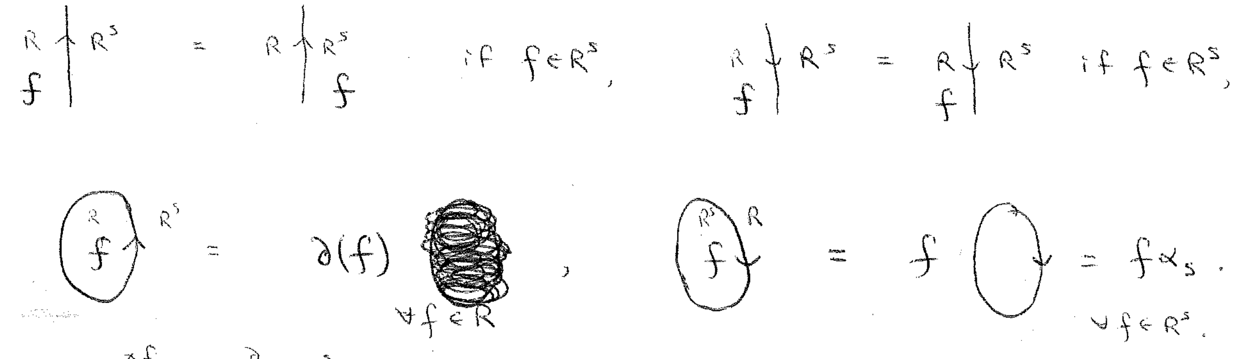
$R \otimes_{R^s} R \quad \Delta(1)$

$\uparrow \qquad \uparrow$

$R \qquad 1$

i.e. multiplication by α_s.

We could also [illegible] insert elements of R^s or R in a region with the appropriate label to represent the multiplication of that map by that element. Then we have the relations



$$\left(R^s \hookrightarrow R \xrightarrow{\times f} R \xrightarrow{\partial} R^s \right.$$
$$\left. = R^s \xrightarrow{\times \partial(f)} R^s \right)$$

The question now is, how can we use such diagrams to describe all bimodule homomorphisms

$$B_s B_t B_u \cdots \longrightarrow B_{s'} B_{t'} B_{u'} \cdots$$

and in particular find all the endomorphisms of

$$B_s B_t B_u \cdots \quad ?$$