Example: 3) If $\mathfrak{g}$ is a complex semisimple Lie algebra, $\underline{h}$ a Cartan subalgebra, $\Phi^+ \subset \Phi$, (W,S) the Weyl group, P the weight lattice, $\hat{Z} = P/\mathbb{Z}\Phi$.

The category $\mathrm{Rep}_f\,\mathfrak{g}$ is $\hat{Z}$-graded. For $z, z' \in \hat{Z}$, let

${}_z\mathrm{Rep}_{z'} = \{ V \in \mathrm{Rep}_f\,\mathfrak{g} \mid V \otimes (\mathrm{Rep}_f\,\mathfrak{g})_{z'} \subset (\mathrm{Rep}_f\,\mathfrak{g})_z \}$.

Of course this is just $(\mathrm{Rep}_f\,\mathfrak{g})_{z-z'}$ but we want to identify these reps with the resulting functors $V \otimes -$.

Then we form a 2-category $R_\mathfrak{g}$ where the objects are elements of $\hat{Z}$ and $\mathrm{Hom}(z', z) = {}_z\mathrm{Rep}_{z'}$.

Let $W_{aff} = W \ltimes \mathbb{Z}\Phi$, a Coxeter gp with $S_{aff} = S \cup \{s_0\}$.

$\hat{Z}$ acts faithfully on (W_{aff}, S_{aff}) via diagram automorphisms.

e.g. W type A_{n-1}: $\hat{Z} = \mathbb{Z}/n\mathbb{Z}$ acting by rotation.

We then have the <u>Satake isomorphism</u>

$[R_\mathfrak{g}] \otimes_{\mathbb{Z}} \mathbb{Z}[v^{\pm 1}] \cong \mathrm{End}_{{}^S\mathcal{H}_{aff}}$(objects of the form $z(S)$ for $z \in \hat{Z}$)

with $[{}_z\mathrm{Rep}_{z'}]$ corresponding to ${}^{z(S)}\mathcal{H}_{aff}{}^{z'(S)}$

e.g. $[{}_0\mathrm{Rep}_0] = [\mathrm{Rep}(G_{ad})]$ corresponds to ${}^S\mathcal{H}_{aff}{}^S = \underline{H}_{w_0}\mathcal{H}_{aff}\underline{H}_{w_0}$

This isomorphism sends classes of simple reps to KL basis elts.

(assume $\underline{h}$ is nice, e.g. faithful & symmetrizable)

Basic fact: if $I \subset J \overset{f}{\subset} S$, then $R^J \subset R^I$ is a Frobenius extension with trace $\partial^J_I : R^I \to R^J$ defined by

$\partial^J_I = \partial_{s_1} \cdots \partial_{s_\ell}$ where $s_1 \cdots s_\ell$ is a reduced expression for $w_J w_I$

Questions: 1) is there an explicit construction of dual bases for ∂^J_I, or even for $\partial^S_\emptyset$ with W finite?

(In type A one gets such a basis using Schubert polynomials.)

2) generators and relations for sSBim (singular Soergel bimodules, defined in an analogous way to SBim)?

Part 6 22/10/13

Singular Soergel bimodules cont'd

Let Bim be the 2-category with:

1) objects = graded rings
2) $Hom(A,B)$ = cat. of graded (B,A)-bimodules with horizontal composition given by tensor product
3) 2-morphisms are bimodule morphisms.

W G $\underline{h}$ defined over $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$ as before, R as before.

For any $I \overset{f}{\subset} S$, set $R^I = R^{W_I}$.

For any $I \subset J \overset{f}{\subset} S$, $R^J \subset R^I$ is a frobenius extension with ∂_I^J as before

sSBim is the smallest 2-subcategory of Bim with:

1) objects R^I for $I \overset{f}{\subset} S$
2) $R^I \in Hom_{sSBim}(R^I, R^J)$ and $R^I \in Hom_{sSBim}(R^J, R^I)$ if $I \subset J \overset{f}{\subset} S$ (induction and restriction bimodules)
3) $Hom_{sSBim}(R^I, R^J)$ is a full ($\oplus$, (n), Karoubian) subcat of $Hom_{Bim}(R^I, R^J)$.

Notation: ${}^I\mathcal{B}^J = Hom_{sSBim}(R^J, R^I)$, a full subcat. of (R^I, R^J)-bimodules

Concretely, the indecomposable objects in ${}^I\mathcal{B}^J$ are the direct summands of bimodules of the form

$$R^{I_1} \otimes_{R^{J_1}} R^{I_2} \otimes_{R^{J_2}} \cdots \otimes_{R^{J_{m-1}}} R^{I_m}(n)$$

for all finitary subsets $I \supset I_1 \subset J_1 \supset I_2 \subset J_2 \supset \cdots I_m \subset J$, $n \in \mathbb{Z}$

Rmk 1) The case of Soergel bimodules is where all $I, J, I_1, \ldots, I_m = \emptyset$ and all $J_1, \ldots, J_{m-1}$ are singletons.

2) (Exercise) If $|W| < \infty$ then the only indecomposable object in ${}^S\mathcal{B}^I$ for any $I \subset S$, up to shift, is R^S $(= R^W)$.

3) (Not obvious) $SBim = End_{sSBim}(R) = {}^{\emptyset}\mathcal{B}^{\emptyset}$.

e.g. $R \otimes_{R^W} R$ is a Soergel bimodule.

(Decategorified, working in SBim is like using only generators $\underline{H}_s$; working in sSBim is like using $\underline{H}_{W_I}$ for all $I \overset{f}{\subset} S$.)

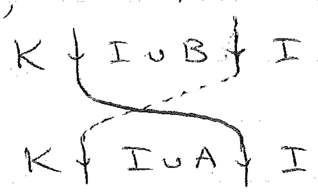
Thm 1) The indecomposable objects in ${}^I\mathcal{B}^J$ are parametrized, up to shift, by $W_I \backslash W / W_J$.

2) $[{}^I\mathcal{B}^J] \xrightarrow[ch]{\sim} {}^I\mathcal{H}^J = \underline{H}_{W_I}\mathcal{H} \cap \mathcal{H}\underline{H}_{W_J}$. (There is again a filtration by support used to define ch.)

3) $[sSBim] \xrightarrow[ch]{\sim} s\mathcal{H}$ (the Schur algebroid), the isom. being compatible with products.

The Frobenius extension $R^J \subset R^I$ gives us morphisms

If I, A, B are pairwise disjoint subsets of S and $K = I \cup A \cup B$ is finitary, then the restriction bimodule $R^I \in (R^K, R^I)$-Bim



can be written in two ways, so we get the isomorphism depicted here.

Fact: these "easy" morphisms generate all morphisms in sSBim. This shows, by comparison with SBim, that sSBim is more natural (easier set of generating morphisms). However, we still don't know relations for these generating morphisms in sSBim, whereas we will see relations for SBim.

Satake isomorphism cont'd

With definitions as on p. 19, suppose $\mathfrak{g} = \mathfrak{sl}_2$.

Then $W_{aff} = \underset{s}{\bullet} \overset{\infty}{-\!-\!-} \underset{t = s_0}{\bullet}$, $\hat{Z} = \{0, 1\}$ acting by $s \leftrightarrow t$.

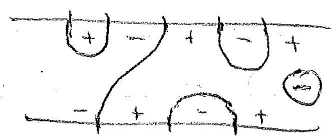
Prop $R_{\mathfrak{sl}_2}$ is $(TL_{\pm})_{\oplus, Kar}$ where $TL_{\pm}$ (signed Temperley-Lieb) has objects m_+, m_-

$$\underset{+\;\;-\;\;+\;\;-\;\;+\;\;-}{\text{—|—|—|—|—|—}} = 5_-$$

alternating signs — label by right-hand edge.

morphisms: isotopy classes of 1-manifolds as before with regions alternately labelled by $\pm$, compatibly with the boundary

e.g. in $\mathrm{Hom}(3_+, 5_+)$



Note: $\mathrm{Hom}(m_\varepsilon, n_{\varepsilon'}) = 0$ unless $\varepsilon = \varepsilon'$ and $m - n$ even.

relations:

$$\textcircled{+}^{\,-} = -2, \qquad \textcircled{-}^{\,+} = -2.$$

We want to compare this with the other categorification of $\bigoplus_{z, z'} {}^{z(S)}\mathcal{H}_{aff}{}^{z'(S)}$, namely sSBim for W_{aff} (note that $S = \{s\}$ and $z(S) = \{t\}$ are exactly the finitary subsets).

Fix $\underline{h} = \mathbb{R}^2$, $\underline{h}^* = \mathbb{R}\alpha_s \oplus \mathbb{R}\alpha_t$. Define $\alpha_s^\vee, \alpha_t^\vee \in \underline{h}$ by

	α_s	α_t
$\alpha_s^\vee$	2	-2
$\alpha_t^\vee$	-2	2

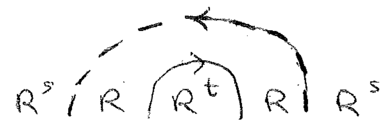
(so they are linearly dependent, but it does no harm).

We want to understand $\overset{sSBim=}{\wedge}\ {}^s\mathcal{B}^t \oplus {}^t\mathcal{B}^s \oplus {}^s\mathcal{B}^s \oplus {}^t\mathcal{B}^t$.

This is generated by ${}_{R^s}R_{R^t}(1) \in {}^s\mathcal{B}^t$ and ${}_{R^t}R_{R^s}(1) \in {}^t\mathcal{B}^s$.

Because of the Frobenius property, these two bimodules are biadjoint.

e.g. The counit ${}_{R^s}R \otimes_{R^t} R_{R^s}(2) \longrightarrow R^s$ is given by



Abbreviated notation:

$R^t \downarrow R \uparrow R^s$ will be denoted $t|s$.

So the above picture becomes $(t)^s$

So we get a functor from isotopy classes of diagrams to sSBim, and we just have to decide the value of the circle.

$$(t)\ s \;=\; (R^t)\ R\ \ R^s \;=\; (\alpha_t) = \partial_s(\alpha_t) = -2.$$

Hence we get a functor

$r: TL_{\pm} \longrightarrow sSBim$ (domain is not graded, codomain is).

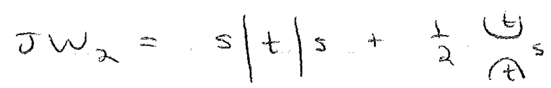
$-|+ \longmapsto t|s$ and vice versa.

Fact: r is an isomorphism on degree-0 morphisms.

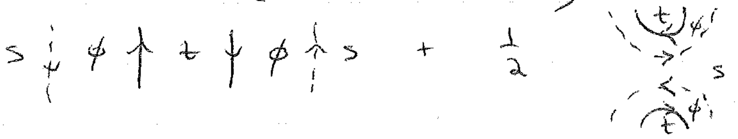
Hence we get the idempotents projecting to the indecomposable singular Soergel bimodules as images of Jones-Wenzl projectors, with $r(JW_m)$ being an idempotent in $\mathrm{End}(\cdots R\otimes_{R^t} R_{R^s}(m))$ ← m factors

[r induces an equivalence between $\mathcal{R}_{sl_2}$ and the degree-zero subcategory of sSBim (some manifestation of geometric Satake).]

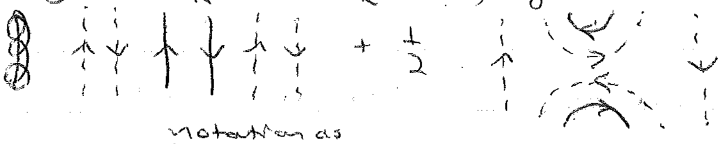
which then gives an idempotent in $\mathrm{End}(\underbrace{R\otimes_{R^{s\text{ or }t}} R\otimes\cdots\otimes R\otimes_{R^s} R}_{= B_{s/t}\cdots B_t B_s}\ (?))$.

Example: 

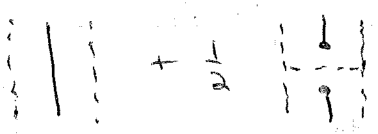
i.e. (un-abbreviated)



Applying $R \otimes_{R^s} (-) \otimes_{R^s} R$, get



i.e. in our ~~formula~~ notation as for the rank-1 setting,



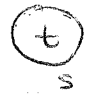
Exercise: use the rank-1 relations to check this is idempotent.

Now consider the general rank-2 case

$W = \langle s, t \mid (st)^m = id \rangle$, $\quad \underline{h} = \mathbb{R}\alpha_s^\vee \oplus \mathbb{R}\alpha_t^\vee$, $\underline{h}^* = \mathbb{R}\alpha_s \oplus \mathbb{R}\alpha_t$

	α_s	α_t
$\alpha_s^\vee$	2	$-2\cos(\pi/m)$
$\alpha_t^\vee$	$-2\cos(\pi/m)$	2

As in the $m = \infty$ case,

 $= -2\cos(\pi/m) = -[2]^{q=\varepsilon}$ for $\varepsilon = e^{2\pi i/m}$.

Moral: TL at $q = \varepsilon$ "controls" Soergel bimodules in rank 2.

One gets a functor $r : TL_{\pm}^{q=\varepsilon} \longrightarrow sSBim$ as before,

but r is no longer surjective on morphisms.

Why? In the K-L basis,

$$\underline{H}_s \underline{H}_t \underline{H}_s \cdots = \underline{H}_{w_0} + \text{lower terms},$$

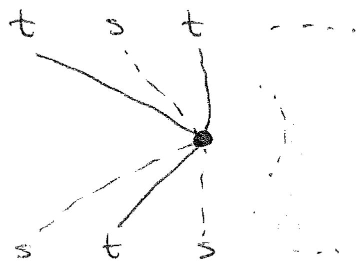
$$\underline{H}_t \underline{H}_s \underline{H}_t \cdots = \underline{H}_{w_0} + (\text{lower terms})'.$$

Exercise: the image of r in $\mathrm{Hom}(\underbrace{B_s B_t \cdots}_{m \text{ times}}, \underbrace{B_t B_s \cdots}_{m \text{ times}})$ is zero,

so we don't get the expected map

$$B_s B_t \cdots \twoheadrightarrow B_{w_0} \hookrightarrow B_t B_s \cdots$$

So we need to introduce a new morphism:



(2m)-valent vertex

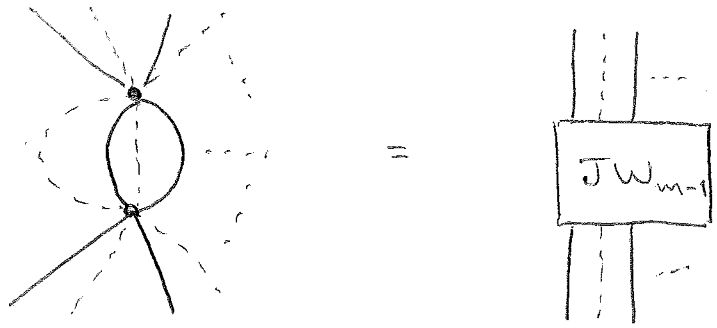
denoting the unique map $f_{s,t}: \underbrace{B_s B_t B_s \cdots}_{m \text{ factors}} \longrightarrow \underbrace{B_t B_s B_t \cdots}_{m \text{ factors}}$

such that $f_{s,t}(1 \otimes 1 \otimes \cdots \otimes 1) = 1 \otimes 1 \otimes \cdots \otimes 1$

and $f_{s,t}$ may be written as projection to followed by inclusion of $B_{w_0} = R \otimes_{R^W} R\,(\ell(w_0))$.

Note that the composition of this with itself must be the Jones-Wenzl projector.

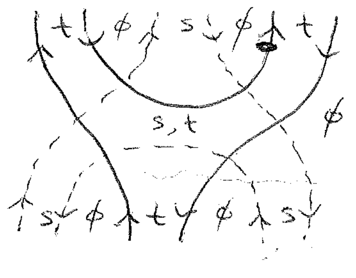
(*)



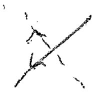
Remark: f_{st} can be given by an explicit but horrible formula, but is nicer when interpreted in singular bimodules:

e.g. $m=3$

$f_{st} =$



Note here:

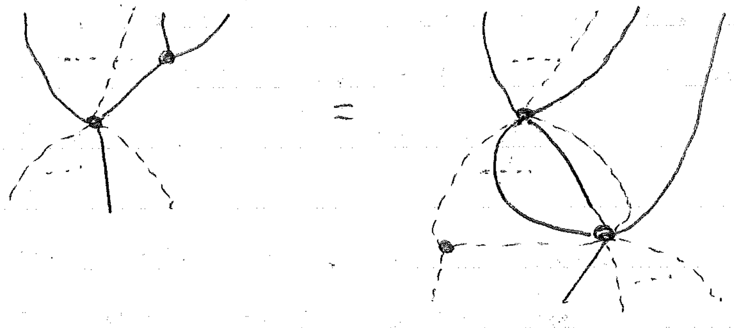


is the nontrivial map

where the crossing is the easy isom of p. 21.

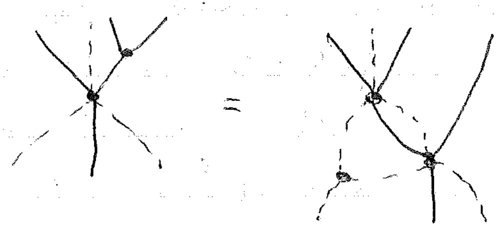
Elias showed that the (2m)-valent vertex satisfies:

(Assoc_2):



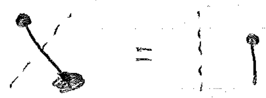
e.g. $m=2$:

$m=3$:



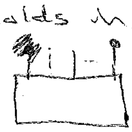
(JW)

Examples $m=2$:



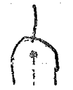
$m=3$:

Sketch of proof: (Assoc_2) is easy once you write it in terms of singular diagrams. (holds in a more general context of Frob. extensions). For (JW), apply



to relation (*) and simplify,

using that the (2m)-valent vertex is "killed by pitchfork" in the sense that it becomes 0 when capped with



:

The final relations needed to present SBim are the Zamolodchikov relations in rank 3.

Motivation: $\mathcal{H}$ has presentation with

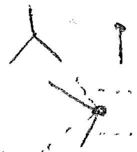
generators $\longleftrightarrow$ $\{s\} \subset S$

relations $\longleftrightarrow$ $\{s\} \subset S$, $\{s,t\} \overset{f}{\subset} S$.

SBim has a presentation with

generating objects $= \{B_s\} \longleftrightarrow \{s\} \subset S$

generating morphisms labelled by

$\{s\} \subset S$,  in each colour, and polynomials

$\{s,t\} \overset{f}{\subset} S$

relations labelled by

$\{s\} \subset S$ (rank 1 case seen before)

$\{s,t\} \overset{f}{\subset} S$ (Assoc_2), (JW)

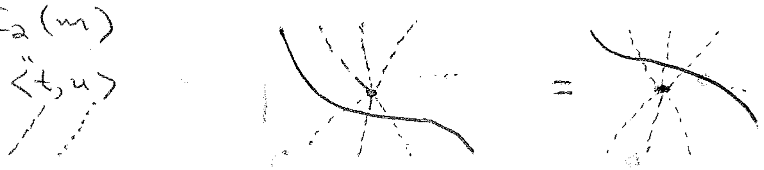
$\{s,t,u\} \overset{f}{\subset} S$ Zamolodchikov

There is one Zam for each $\{s,t,u\} \overset{f}{\subset} S$.

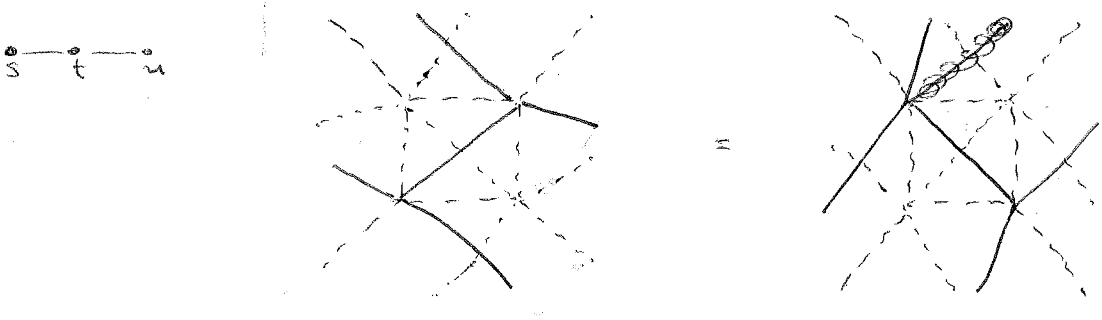
$A_1 \times I_2(m)$

$\langle s \rangle$ $\langle t,u \rangle$

solid



A_3



(front and back of the Coxeter complex regarded as a coloured triangulation of the sphere)

B_3 defined similarly.

Caution: The H_3 Zam doesn't hold on the nose, instead LHS − RHS is a sum of simpler diagrams with undetermined coefficients.

<u>Thm</u> (Elias – W.) This is a presentation for SBim.