Part 07 25/10/13

$\mathcal{H}$ has a presentation $\left\langle H_s, s \in S \,\middle|\, \begin{array}{l} H_s^2 = H_{id} + (v^{-1}-v)H_s \\ \underbrace{H_sH_t\cdots}_{m_{s,t} \text{ terms}} = \underbrace{H_tH_s\cdots}_{m_{s,t} \text{ terms}} \end{array} \right\rangle$,

and starting from this we have to work to get a basis of $\mathcal{H}$.
A key point is Matsumoto's theorem (proved first by Steinberg) that any two reduced expressions can be related by braid relations, so H_w is well-defined as $H_{s_1}\cdots H_{s_m}$ for a reduced expression $w = s_1\cdots s_m$.
One step up the categorical ladder, to Soergel bimodules, we have to consider the "paths between different paths" between reduced expressions, and these are provided by the Zamolodchikov relations. This helps to explain why it is so hard to describe bases for homs between Soergel bimodules.

We start with Soergel's hom formula.
Khovanov's philosophy: categorification casts two shadows,
1) a positive basis (e.g. KL basis, canonical basis, ...)
2) a bilinear form (or sesquilinear over $\mathbb{Z}[v^{\pm1}]$) $([A],[B]) = \dim \mathrm{Hom}(A,B)$, (or graded dim) in some sense.

Often relations can be guessed from knowledge of the form.

Consider the form on $\mathcal{H}$ given by $(h,h') = \varepsilon(h'\omega(h))$
where ω is the anti-involution defined by $\omega(v) = v^{-1}$, $\omega(H_x) = H_x^{-1}$
and $\varepsilon(\sum a_x H_x) = a_{id}$.

Exercises 1) $\forall h, h' \in \mathcal{H}$, $\varepsilon(hh') = \varepsilon(h'h)$.
2) $(ph, qh') = \bar{p}q(h,h')$ $\forall p,q \in \mathbb{Z}[v^{\pm1}]$.
3) $(\underline{H}_s h, h') = (h, \underline{H}_s h')$
$(h\underline{H}_s, h') = (h, h'\underline{H}_s)$ (corresponding to biadjointness of B_s)
4) $(H_x, H_y) = \delta_{xy}$.
5) $(\underline{H}_x, \underline{H}_y) \in \delta_{xy} + v\mathbb{Z}[v]$. (corresponding to Soergel's conjecture)

Suppose the action of W on $\underline{h}$ is reflection faithful, meaning faithful and also $\dim \underline{h}/\underline{h}^x = 1 \iff x$ is a conjugate of an element of S.

Also suppose $\underline{h}$ is over an infinite field.

Recall the $\mathbb{Z}[v^{\pm 1}]$-alg. isom ch: $[\mathrm{SBim}] \xrightarrow{\sim} \mathcal{H}$ sending $[B_s]$ to $\underline{H}_s$.

<u>Soergel's hom formula:</u> $\forall\, B, B' \in \mathrm{SBim}$,

$\mathrm{Hom}^\bullet(B, B')$ is free as a left (or right) R-module of graded rank $(\mathrm{ch}(B), \mathrm{ch}(B'))$.

homs of all degrees

Notation: $\underline{x} = (s_{i_1}, \dots, s_{i_m})$ is an expression for $x = s_{i_1} \cdots s_{i_m} \in W$.

Say $\underline{x}$ is a <u>rex</u> (reduced expression) if $l(x) = m$.

$H_{\underline{x}}$ means $H_{s_{i_1}} \cdots H_{s_{i_m}}$ ($= H_x$ if $\underline{x}$ is a rex).

$\underline{H}_{\underline{x}}$ means $\underline{H}_{s_{i_1}} \cdots \underline{H}_{s_{i_m}}$ (usually not equal to $\underline{H}_x$)

$B_{\underline{x}} = B_{s_{i_1}} \cdots B_{s_{i_m}}$ (a Bott–Samelson bimodule, usually not isom to B_x)

Bott–Samelson special case of Soergel's hom formula:

$$\text{graded rank of } \mathrm{Hom}^\bullet(B_{\underline{x}}, B_{\underline{x}'}) = (\underline{H}_{\underline{x}}, \underline{H}_{\underline{x}'}).$$

By biadjointness, this is implied by the statement that

$$\text{graded rank of } \mathrm{Hom}^\bullet(B_{\underline{x}}, R) = \varepsilon(\underline{H}_{\underline{x}}).$$

<u>Deodhar's defect formula:</u>

A subexpression of $\underline{x} = (s_1, \dots, s_m)$ is determined by a sequence $\underline{e} = (e_1, \dots, e_m)$, $e_i \in \{0,1\}$

The <u>Bruhat stroll</u> is the sequence

$$x_0 = id,\ x_1 = s_1^{e_1},\ x_2 = s_1^{e_1} s_2^{e_2},\ \dots,\ x_m = s_1^{e_1} \cdots s_m^{e_m} =: \underline{x}^{\underline{e}}.$$

We decorate $\underline{e}$ with "U" (up) or "D" (down) as follows:

if $x_{i-1} s_i > x_{i-1}$, then U for e_i

if $x_{i-1} s_i < x_{i-1}$, then D for e_i (regardless of whether $e_i = 0$ or 1)

Example: $\underline{x} = sts$ with $s \neq t$

$\underline{e} = 000 \rightsquigarrow$ U0 U0 U0
$\underline{e} = 101 \longrightarrow$ U1 U0 D1 } both with $\underline{x}^{\underline{e}} = id$

$\underline{e} = 100 \rightsquigarrow$ U1 U0 D0
$\underline{e} = 001 \longrightarrow$ U0 U0 U1 } both with $\underline{x}^{\underline{e}} = s$.

The <u>defect</u> $d(\underline{e})$ = number of U0's − number of D0's.

<u>Exercise</u> $\underline{H}_{\underline{x}} = \sum_{\underline{e}} v^{d(\underline{e})} H_{\underline{x}^{\underline{e}}}$.

Hence $\varepsilon(\underline{H}_{\underline{x}}) = \sum_{\substack{\underline{e} \\ \underline{x}^{\underline{e}} = id}} v^{d(\underline{e})}$.

Using generators and relations, we will construct an explicit map

$$\left\{ \begin{array}{l} \underline{e} \text{ subexpression of } \underline{x} \\ \text{s.t. } \underline{x}^{\underline{e}} = id \end{array} \right\} \xrightarrow{LL} \mathrm{Hom}^{\bullet}(B_{\underline{x}}, R)$$

"light leaves"
(terminology of Libedinsky)

matching defect with degree

Libedinsky's theorem: $\{ LL(\underline{e}) \mid \underline{x}^{\underline{e}} = id \}$ is a basis of $\mathrm{Hom}^{\bullet}(B_{\underline{x}}, R)$
(under the assumptions of Soergel's hom formula) as a graded R-module.

Elias-W: this also holds more generally in the diagrammatic category.
(The proof is probably harder, but more "formal".)