Basic Q's in representation theory

Friday 4/10/13

1) simple modules for $S_n$ in char $p$? (oldest Q in rep theory?) open

$p > \sqrt{n}$ : James conjecture (1990) $\longleftarrow$ false for $S_N$ with $N \sim 400\,000$.

$p \leq \sqrt{n}$ : no conjecture.

2) $\mathfrak{g}$ ~~$\supset \mathfrak{b} \supset \mathfrak{h}$~~ complex semi-simple Lie algebra.

Describe Rep $\mathfrak{g}$ ... $\longrightarrow$ harmonic analysis

$\longrightarrow$ Langlands program ...

~~$\mathfrak{b} \subset \mathfrak{g}$ Borel sub algebra.~~

characters of simple highest weight modules? (i.e. loc. ~~fo~~ finite) $\mathfrak{b} \subset \mathfrak{g}$ Borel

Kazhdan-Lusztig conjecture (1979) proof by Brylinski-Kashiwara and Beilinson-Bernstein (1981).

(new) proof by EW using Soergel bimodules

answer in terms of $P_{x,w}(1) \longleftarrow$ Kazhdan-Lusztig polynomial.

Jantzen conjecture (1979): explains why polynomials appear proof by Beilinson-Bernstein in 1990. (work in progress)

Kazhdan-Lusztig positivity conjecture: $P_{x,w} \in \mathbb{N}[q]$ for any $(W,S)$ Coxeter system

(KL: proof for Weyl and affine Weyl groups 1980.)

3) $G/\mathbb{F}_p$ split reductive group (eg $GL_n(\mathbb{F}_p)$).

simple modules for $G(\mathbb{F}_p)$ $\begin{cases} \text{in char } 0 \text{ (Deligne-Lusztig theory, ... "complete" answer)} \\ \text{in char } \ell \neq p \text{ (mod. DL theory, Broué's conjecture, ...)} \\ \text{in char } p \end{cases}$

$\longleftrightarrow$ (thms of Steinberg).

simple rational reps of $G$

$p \geq h$ (Coxeter number) (e.g. $h = n$ for $GL_n$) $\rightsquigarrow$ Lusztig's conjecture ( ? false for $h \leq p \leq ?\exp(h)$ )

have counterexamples, but exact ranges are not yet clear.

$p \leq h$ $\rightsquigarrow$ no conjecture.

## $\underline{\mathcal{O}}$ : a very useful microcosm for the ideas of this course

historically the source of many ideas: translation functors, categorification, connections to geometry / perverse sheaves, hidden gradings, Koszulity, ...

$\mathfrak{g}$: semi-simple Lie algebra / $\mathbb{C}$

$\cup$

$\mathfrak{h}$ : Cartan subalgebra $\qquad \underset{\text{simple positive roots}}{\Delta \subset R^+ \subset R \subset \mathfrak{h}^*}$

$\rightsquigarrow \quad \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ .$

(BGG) $\mathcal{O}$ : full subcat of $\mathfrak{g}$-modules which are $\begin{cases} \text{f.g.} / \mathfrak{g} \\ \text{loc. fin.} / \mathfrak{b} \\ \text{weight} / \mathfrak{h} . \end{cases}$

Eg: given $\lambda \in \mathfrak{h}^*$ form $\Delta(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ "Verma module".

$\Delta(\lambda) \twoheadrightarrow L(\lambda)$ unique irred. quot.

$$\begin{array}{ccc} \{\text{simple objects in } \mathcal{O}\}_{/\cong} & \longleftrightarrow & \mathfrak{h}^* \\ \Psi & & \Psi \\ L(\lambda) & \longleftrightarrow & \lambda \end{array}$$

Eg: $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e \qquad L(-2) \hookrightarrow \Delta(0) \twoheadrightarrow L(0) = \mathbb{C}$

$\uparrow \qquad \uparrow$

simple modules

Write $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ and let $W$ = Weyl group $\curvearrowright \mathfrak{h}^*$.

$x \cdot \lambda := x(\lambda+\rho) - \rho .$ "dot action".

Harish-Chandra isomorphism: $U(\mathfrak{g}) \supset \underset{\text{centre}}{Z} \cong S(\mathfrak{h})^{(W\cdot)}$ ("$\cdot$ invariants").

Set $Z^+ := \mathrm{Ann}_Z \, \mathbb{C}$ ← trivial module.

$\mathcal{O}_0$ := full subcat of $M \in \mathcal{O}$ s.t. $(Z^+)^m \cdot M = 0$ for $m \gg 0$.

$\cap$ ~~block~~ principal block.

$\mathcal{O}$

Eg: $\mathfrak{g} = \mathfrak{sl}_2$. $\mathcal{O}_0 = \langle L(-2), L(0) \rangle_{\text{weight modules}}$

$= \langle \Delta(-2), \Delta(0) \rangle$.

Exercise: describe all objects in $\mathcal{O}_0$ for $\mathfrak{sl}_2(\mathbb{C})$.

Facts about $\mathcal{O}_0$: simple objects $\longleftrightarrow W$

$$\begin{array}{ccc} \text{simple objects} & \longleftrightarrow & W \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ L(w \cdot 0) & & w \end{array}$$

finite length, enough projectives, BGG reciprocity: ~~$(P(x\cdot 0) : \Delta(y\cdot 0)) = (\Delta(y\cdot 0) : L(x\cdot 0))$~~.

denote by $P(w\cdot 0)$ the projective cover of $L(w\cdot 0)$.

We have $[\mathcal{O}_0] = \bigoplus \mathbb{Z}[L(w\cdot 0)] = \bigoplus \mathbb{Z}[\Delta(w\cdot 0)] = \bigoplus \mathbb{Z}[P(w\cdot 0)]$.

BGG reciprocity: $(P(x\cdot 0) : \Delta(y\cdot 0)) = [\Delta(y\cdot 0) : L(x\cdot 0)]$.

"tailor made to solve the multiplicity question".

There exist "exact wall-crossing" functors $\vartheta_s \circlearrowright \mathcal{O}_0$ s.t.

$$\vartheta_s([\Delta(w\cdot 0)]) = [\Delta(w\cdot 0)] + [\Delta(ws\cdot 0)].$$

Moreover, $P(x\cdot 0)$ is a summand of

$$\vartheta_s \vartheta_t \cdots \vartheta_u (\Delta(0)) \quad \text{if } st\ldots u \text{ is a red. exp. for } x^{-1}.$$

$\uparrow$

KL conjecture: understand the decomposition of these modules.

Ex: For $\mathfrak{sl}_2(\mathbb{C})$ show that $\vartheta_s \Delta(0) = P(s\cdot 0)$. "easy".

Soergel: one can completely understand morphisms ~~betwe~~

$\mathrm{Hom}(\vartheta_s \dots \vartheta_t, \vartheta_{s'} \dots \vartheta_{t'})$ in terms of Soergel bimodules (commutative algebra).

Khovanov-Elias, Libedinsky, E, EW: one can present this category

via explicit generators and relations

$\rightsquigarrow$ category theory, higher algebra, diagrammatics, homotopy ...

(answers a Q of Gelfand ~1979).

KL conjecture becomes a question (very difficult) about the existence of certain idempotents.

Elias-W: one can use ideas from Hodge theory (de Cataldo-Migliorini)

to prove the existence of ~~certain~~ these idempotents

(hard Lefschetz + Hodge-Riemann bilinear relations).

These deep thms fail modulo $p$ $\rightsquigarrow$ ~~what remains?~~

~~p-canonical basis~~

understanding their failure should be the key to many problems in modular rep theory.