String diagrams:

Consider two categories $\mathcal{A}$ and $\mathcal{B}$ and functors

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "\hat{F}"] & \mathcal{B} \arrow[l, shift left, "F"] \end{tikzcd}$$

Assume: $\hat{F} \dashv F$ $\iff$ $\mathrm{Hom}(\hat{F}, \hat{F}) = \mathrm{Hom}(\mathrm{id}_{\mathcal{A}}, F\hat{F})$

is left adjoint to

$$\underset{\mathrm{id}}{\Psi} \qquad \underset{\eta}{\Psi} \quad \text{unit}$$

$$\mathrm{Hom}(F, F) = \mathrm{Hom}(\hat{F}F, \mathrm{id})$$

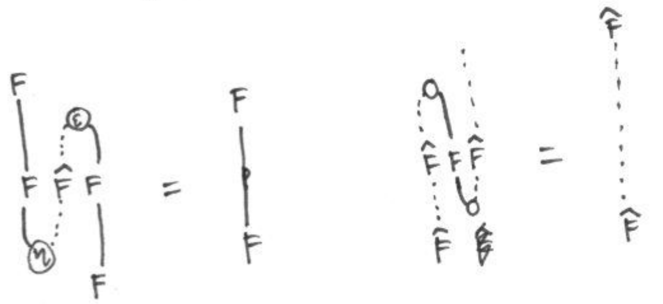
$$\underset{\mathrm{id}}{\Psi} \qquad \underset{\varepsilon}{\Psi} \quad \text{counit}$$

such that

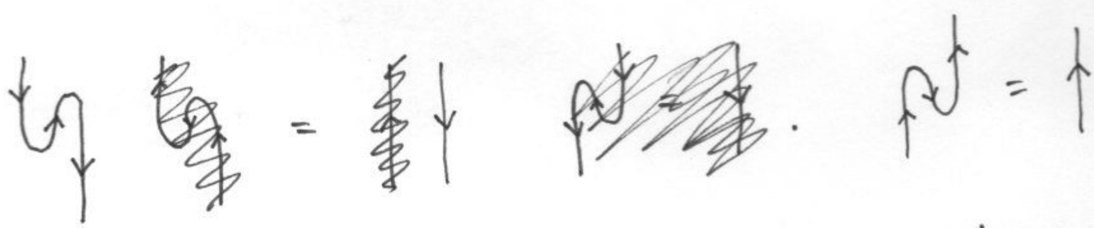
$$F \xrightarrow{\eta F} F\hat{F}F \xrightarrow{F\varepsilon} F \quad \text{identity}$$

$$\hat{F} \xrightarrow{\eta \hat{F}\eta} \hat{F}F\hat{F} \xrightarrow{\varepsilon \hat{F}} \hat{F} \quad \text{identity.}$$

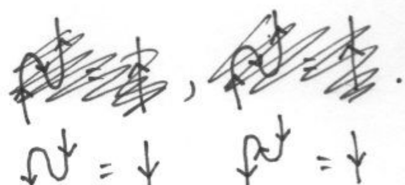
We can depict this graphically as follows:



If we draw the identity on F as $\downarrow$ and identity on $\hat{F}$ as $\uparrow$ then we have:

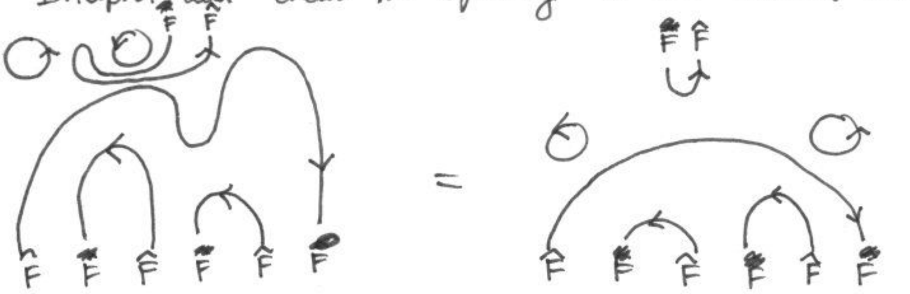


Now suppose that ~~EA~~ $F \dashv \hat{F}$ also. Then



Biadjointness + axioms of 2-categories: any oriented planar diagram determines a well-defined morphism of functors.

Exercise: Interpret and check the equality in the following diagram



Now suppose that $A \subset B$ is an inclusion of rings and $\mathcal{A} = A\text{-mod}$, $\mathcal{B} = B\text{-mod}$.

$$A\text{-mod} \underset{\mathrm{res} = F \,\cong\, {}_A B_B}{\overset{{}_B B \otimes_A \,\cong\, \hat{F} = \mathrm{ind}}{\rightleftarrows}} B\text{-mod}$$

Basic fact: equivalence of 2-cats: right continuous functors on module categories (i.e. right exact and commute with direct limits)

$\cong$

bimodules.

Verbally: any right continuous functor F is given by tensoring with a bimodule B_F

and for right continuous F, G $\mathrm{Hom}_{\text{Functors}}(F, G) = \mathrm{Hom}_{\text{bimodules}}(B_F, B_G)$.

unit: ${}_A A_A \longrightarrow {}_A B \otimes_B B_A = {}_A B_A$ (inclusion)

counit: ${}_A B \otimes_A B_A \longrightarrow {}_A B_A$ (multiplication)

Ex: Check Zig Zag relations:

$$\begin{array}{cc} {}_A B_B & b \\ \uparrow & \uparrow \\ {}_B B \otimes_B B \otimes_A B_B & 1 \otimes 1 \otimes b \\ \uparrow & \uparrow \\ {}_A B_B & b \end{array}$$

Def: $A \subset B$ is a Frobenius extension if $\mathrm{res} \dashv \mathrm{ind}$ (extension also carries the data of the adjunction)

i.e. want maps $\partial : B \longrightarrow A$ (A-bimodules) counit

$\Delta : B \longrightarrow B \otimes_A B$ (of B-bimodules) (unit) satisfying ...

$\mathrm{Hom}_A(B,-)$ has a right adjoint $\Rightarrow$

① $\Rightarrow$ $\mathrm{Hom}_A(B,-)$ is right exact $\Rightarrow$ B is projective as an A-module.

② $\Rightarrow$ $\mathrm{Hom}_A(B,-)$ commutes with injective direct limits $\Rightarrow$ B is f.g. as an A-module.

In fact, ∂ determines the adjunction: (see Rouquier, 2KMalgebras, §2.3.1)

Claim: suppose B satisfies ① and ② and that $\partial : B \to A$ is a map of (A,A)-bimodules such that

$b \mapsto (b' \mapsto \partial(b'b))$ gives an isomorphism $B \xrightarrow{\sim} \mathrm{Hom}_A(B,A)$.

Then ∂ is the counit in a unique adjunction making $A \subset B$ a Frob extension.

Suppose B is a free A-module, then we can describe Δ explicitly.

Let $e_1,\dots,e_m$ be a basis and $e_1^*,\dots,e_m^*$ a dual basis wrt $(b,b') \mapsto \partial(bb')$

then $\Delta(1) = \sum e_i \otimes e_i^*$.

Exercise: Check that $b\Delta(1) = \Delta(1)b$ $\forall b \in B$. Hence Δ is indeed a map of B-bimodules.

All of Soergel bimodule theory is built around Frobenius extensions!

Today we do the simplest case:

$s \circlearrowright \mathfrak{h}$ is a reflection, $R = S(\mathfrak{h}^*)$.
$\alpha_s \in \mathfrak{h}^*$ equation for fixed hyperplane.

Claim: $R^s \subset R$ is a Frobenius extension with

$$\partial(f) = \frac{f - sf}{\alpha_s}.$$

Proof: $\rightarrow$ ∂ is clearly a map of R^s-bimodules.

$\rightarrow$ $R \cong R^s \oplus \alpha_s R^s$ and $\partial(\alpha_s) = \dfrac{\alpha_s - (-\alpha_s)}{\alpha_s} = 2$. $s(\alpha_s^2) = \alpha_s^2$

$\Rightarrow \partial(\alpha_s^2) = 0$.

$\partial(1) = 0$

Hence $e_1 = 1$ $\quad e_1^* = \frac{1}{2}\alpha_s$

$e_2 = \alpha_s$ $\quad e_2^* = \frac{1}{2}$ are dual bases.

Hence $\Delta = \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$.

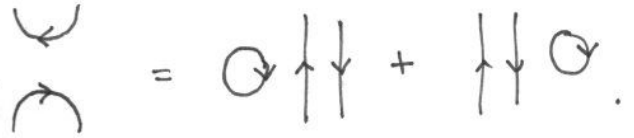
We can add some relations to our graphical calculus:

$\partial_s(f)$ =  s (*$_1$) $\quad R^s \hookrightarrow R \xrightarrow{\partial} R^s$

α_s =  (*$_2$) $\quad R \xrightarrow{\Delta} R \otimes_{R^s} R \xrightarrow{m} R$

$\frac{1}{2}\alpha_s + \frac{1}{2}\alpha_s = \alpha_s$  1

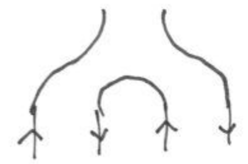
$1 \otimes 1 \mapsto \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$ 

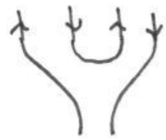
 (*$_3$)

Now to Soergel bimodules: B_s = ↑↓. $B_s = R \otimes_{R^s} R(1)$ = ↑↓ (ignore shifts)

Consider the following maps:

$B_s \to R$, $\quad f \otimes g \mapsto fg$  

$B_s B_s \to B_s$, $\quad f \otimes g \otimes h \mapsto f \otimes \partial(g) h$  

$B_s \to B_s B_s$, $\quad f \otimes g \mapsto f \otimes 1 \otimes g$  

$R \to B_s$, $\quad 1 \mapsto \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$  

<u>Relations:</u> B_s is a Frobenius object: "algebra / coalgebra"

 etc. (all obvious from singular relations)

Non-standard relations:

$$f \,\big| \;=\; \big|\, s(f) + \partial_s(f).$$

True for $f \in R^s$ (obvious).

True for α from $(*_3)$ above.

$(*_2) \Rightarrow \; = \alpha_s$

$\quad (f) = \partial_s(f) \quad$ (by $(*_1)$).

Thm (Libedinsky, Elias-Khovanov) These are all relations.