$X \subset \mathbb{P}^n$ $\qquad$ $H := H^\bullet(X;\mathbb{R})$ (deRham or singular cohom.)

connected, smooth, projective, $\dim_{\mathbb{C}} = d$.

<u>Hard Lefschetz thm:</u> $\omega = c_1(\mathcal{O}(1)) \in H^2$. "ample class"

$\omega^i : H^{d-i} \longrightarrow H^{d+i}$ is an isomorphism. $\forall i \geq 0$.

$$\Updownarrow$$

there exists an $\mathfrak{sl}_2(\mathbb{R})$-action on H with $h(x) = (d - \deg(x))$
$e(x) = \omega x$.

(then f is unique).

Eg: $d=4$

$$\begin{tikzcd}
8 & \mathbb{R} & & & & \\
6 & \mathbb{R} \arrow[u, "s"'] & \oplus & \mathbb{R} & & \\
4 & \mathbb{R} \arrow[u, "s"'] & \oplus & \mathbb{R} \arrow[u, "s"'] & \oplus & \textcircled{$\mathbb{R}$}_{p^4} \\
2 & \mathbb{R} \arrow[u, "s"'] & \oplus & \textcircled{$\mathbb{R}$}_{p^2} \arrow[u, "s"'] & & \\
0 & \textcircled{$\mathbb{R}$}_{p^0} \arrow[u, "s"'] & & & &
\end{tikzcd}$$

For $i \geq 0$ set $\quad P^{d-i} := \ker \omega^{i+1} \subset H^{d-i}$ $\quad$ ("lowest weight space")
$\ker f$.

Assume $H^{p,q}(X) \neq 0 \Rightarrow p = q$. "generated by algebraic cycles". (AC).

<u>Hodge-Riemann bilinear relations:</u>

For $i \leq d$ define $\forall x, y \in H^i$

$$(x,y)^i_\omega := \langle x, \omega^{d-i} y \rangle.$$

The restriction of $(-,-)^i_\omega$ to $P^i \subset H^i$

is $(-1)^{i/2}$-definite.

<u>Rmk:</u> 1) $H^m = 0$ for odd $m \Rightarrow (-1)^{i/2}$ makes sense.

2) can drop (AC) but need to take Hodge numbers into account.

<u>On board:</u>

Eg: $X = Gr(3,6)$. $d = 9$.

i	$\dim H^i$
18	1
16	1
14	2
12	3
10	3
8	3
6	3
4	2
2	1
0	1

$$\begin{tikzcd}
\mathbb{R} & & \\
\mathbb{R} \arrow[u] & & \\
\mathbb{R} \arrow[u] & \mathbb{R} & \\
\mathbb{R} \arrow[u] & \mathbb{R} \arrow[u] & \mathbb{R} \\
\mathbb{R} \arrow[u] & \mathbb{R} \arrow[u] & \mathbb{R} \arrow[u] \\
\mathbb{R} \arrow[u] & \mathbb{R} \arrow[u] & \mathbb{R} \arrow[u] \\
\mathbb{R} \arrow[u] & \mathbb{R} \arrow[u] & \mathbb{R} \arrow[u] \\
\mathbb{R} \arrow[u] & \mathbb{R} \arrow[u] & \\
\mathbb{R} \arrow[u] & & \\
\mathbb{R} \arrow[u] & &
\end{tikzcd}$$

Hodge: positivity considerations give hard Lefschetz. $\boxed{\text{extended by decataldo-Migliorini}}$

Chern: use Hodge $*$ to give f, check $\mathfrak{sl}_2$ rel$^{\underline{n}}$s $\Rightarrow$ hard Lefschetz.

Grothendieck: f should exist as an algebraic cycle on $X \times X$

$$\Uparrow$$

Hodge conjecture.

Rmk: in algebraic settings, direct constructions of f seems difficult.

(hard Lefschetz, ~~etc~~ HR)

Generalizations: 1) everything remains true for choices of d ample classes $\omega_1, \dots, \omega_d$.

(ref?)

$$H^0 \xrightarrow{\omega_1} H^2 \xrightarrow{\omega_2} H^4 \longrightarrow \dots \xrightarrow{\omega_d} H^{2d}.$$

2) For any ample ω, consider $\mathfrak{sl}_2(\mathbb{R})_\omega \subset \mathrm{End}(H)$.

Then $\mathrm{Lie}\, X := \langle \mathfrak{sl}_2(\mathbb{R})_\omega \rangle \underset{\substack{\uparrow \\ \text{Lie subalgebra}}}{\subset} \mathrm{End}(H)$.

(L , Lunts)

Then: $\mathrm{Lie}\, X$ is reductive.

Eg: $\mathrm{Lie}\, \mathbb{P}^n X = \mathfrak{sl}_2(\mathbb{R})$, Lie (abelian var) $= \bigoplus \mathfrak{sl}_2(\mathbb{R})$, $\mathrm{Lie}\, G/B = \mathrm{Aut}(H^*(G/B), \langle -,- \rangle)$.

The coinvariant algebra: $\underset{\substack{\text{complex} \\ \text{reductive}}}{G} \supset \underset{\text{Borel}}{B} \supset \underset{\substack{\text{max} \\ \text{torus}}}{T} \supset \underset{\substack{\text{compact} \\ \text{torus}}}{K}$

$\Phi \subset (\mathrm{Lie}\, K)^*$ root system

$\Phi^\vee \subset (\mathrm{Lie}\, K)$ coroots

$\rightsquigarrow$ Φ_+^{Φ}, $\Phi_+^\vee$ pos. roots.

W Weyl group

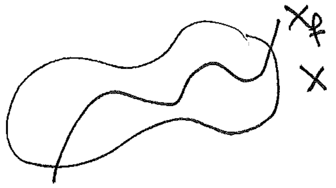
$\circlearrowright$ $\circlearrowright$ $\circlearrowright$

K $\quad \mathrm{Lie}\, K \quad S((\mathrm{Lie}\, K)^*) =: R$

$\uparrow$ deg 2

Borel isomorphism: $H^\bullet(G/B) \cong R/(R_+^W)$.

in particular, $H^2(G/B) = (\mathrm{Lie}\, K)^*$.



$$\begin{tikzcd}[row sep=tiny] & \text{Lefschetz (monodromy)} \\ \text{History:} \arrow[ur] \arrow[dr] & \\ & \text{Hodge (positivity).} \end{tikzcd}$$

Lefschetz (1930's): reg $:= \{f \in \Gamma(X, \mathcal{O}(1)) \mid X_f := \{x \in X \mid f(x) = 0\}$ is smooth$\}$

$\bigcap$ Zariski open

$\Gamma(X, \mathcal{O}(1))$

For ~~[illegible]~~ there exists a local system $\mathcal{L}^i$ on reg $\mathcal{L}^i_f := H^i(X_f)$.

Fix $f \in$ reg, $\pi_1(\text{reg}) := \pi_1(\text{reg}, f) \circlearrowright H^i(X_f)$.

Weak Lefschetz thm $\Longrightarrow$ $\pi_1(\text{reg}) \circlearrowright H^i(X_f)$ trivial unless $i = d-1$.

Lefschetz:

$$\begin{tikzcd} & H^{d-1} \arrow[rr, "\omega"] \arrow[dl, "s"'] \arrow[dr, hook, "\text{res}^*"] & & H^{d+1} & & \text{(crucial case)} \\ H^{d-1}(X_f)^{\pi_1(\text{reg})} \arrow[rr, hook] \arrow[rrrr, bend right, "c"'] & & H^{d-1}(X_f) \arrow[rr, two heads] \arrow[ur, two heads, "\text{res}_*"] & & H^{d-1}(X_f)_{\pi_1(\text{reg})} \arrow[ul, "s"'] & \end{tikzcd}$$

Hence ω iso $\Longleftrightarrow$ c iso $\Longleftrightarrow$ $H^{d-1}(X_f)^{\pi(\text{reg})} \overset{\oplus}{\subset} H^{d-1}(X_f)$ as $\pi_1(\text{reg})$-modules.

Lefschetz: "obvious" that $\pi_1(\text{reg}) \circlearrowright H^{d-1}(X_f)$ is semi-simple.

Delignes: proof of Weil conjectures $\Uparrow$. But not true over $\mathbb{Z}$.

Rmk: (Deligne) image of $\pi_1(\text{reg})$ in $\text{Aut}(H^{d-1}(X_f), \langle -,- \rangle_{\text{Poinc}})$

is finite (in which case is ADE Weyl group) or Zariski dense.

Verbal: it seems likely that for rep theory and "small" embeddings

the ADE case will occur.

Perhaps mention arithmetic groups?

~~[illegible]~~

λ ~~ample~~ $\in H^2$ ample $\Longleftrightarrow$ $\langle \lambda, \alpha_s^\vee \rangle > 0 \quad \forall \quad \alpha^\vee \in \Phi_+^\vee$.

Hodge theory: $\quad \lambda : H^\bullet(G/B) \longrightarrow H^{\bullet+2}(G/B)$

$\lambda : IH^\bullet(\overline{BwB}/B) \longrightarrow IH^\bullet(\overline{BwB}/B)$ satisfies hard Lefschetz and HR.

$\uparrow$

intersection cohom.

Elias - W: algebraic proof $\Longrightarrow$ algebraic proof of KL conjecture for highest wt LieG-modules.

works for any Coxeter group (W,S) (with Lie K replaced by a suitable representation) $\Longrightarrow$ proof of KL positivity conjecture (1979).

eg: hL and HR for coinvariants for H_3, H_4.

Q: → Proof uses technology of Soergel bimodules (>100 pgs)
Can one give a direct proof for coinvariant ring (McDaniel, Wachs)
Even classical type has no direct proof.

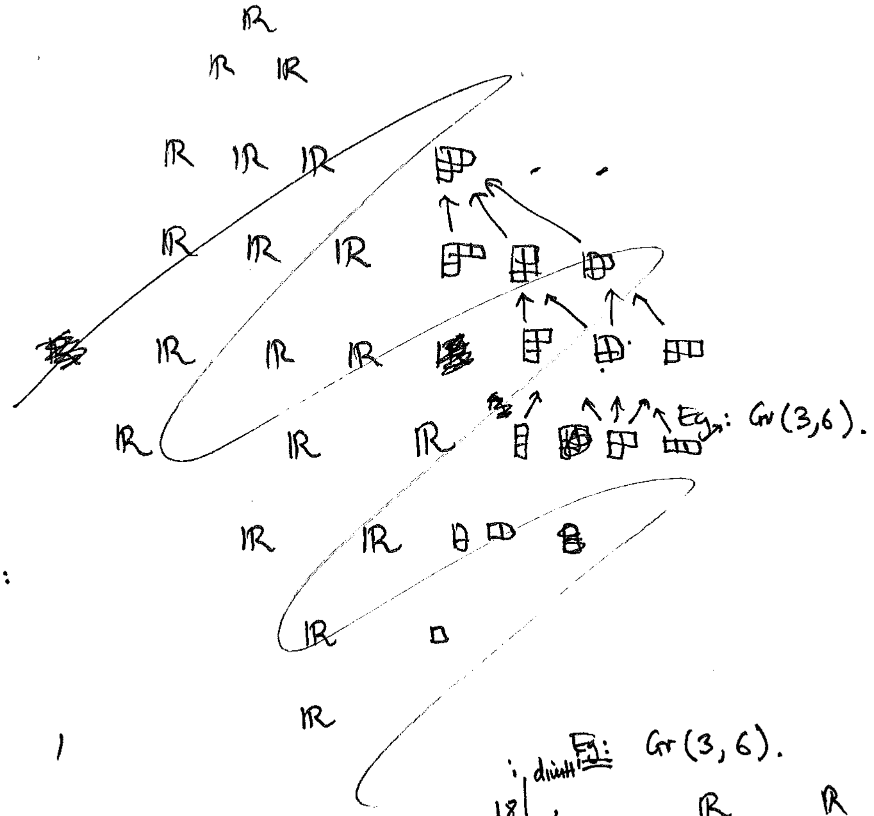
→ What are the signatures of Lefschetz forms for non-dominant λ?

Observation (Soergel): $\quad \mathrm{Lie}\, X_w := \langle \mathfrak{sl}_2(\mathbb{R})_w \rangle_{\text{dominant } w} \subset \mathrm{End}(H\cancel{*(}\, IH(X_w)).$

$IH^*(X_w)$ is a simple module over Lie $\overline{BwB/B}$.

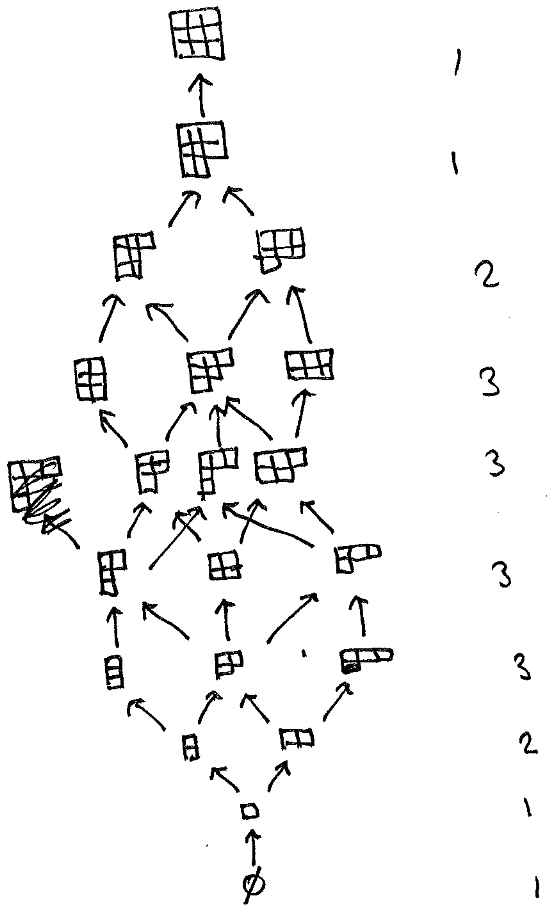
$\longrightarrow$ Is it true that $\cancel{IH^*}$ Lie $= \mathrm{Aut}(IH(\overline{BwB}/B), \langle -,- \rangle_{\text{Poinc}})$?

→ Can one give a Lefschetz style proof? Would be interesting for modular rep theory.



ON BOARD
AT START

In basis of Schubert classes:



Eg: Gr(3,6).

i	dim H	
18	1	$\mathbb{R}$
16	1	$\mathbb{R}$
14	2	$\mathbb{R}^2$
12	3	$\mathbb{R}^3$
10	3	$\mathbb{R}^3$
8	3	$\mathbb{R}^3$
6	3	$\mathbb{R}^3$
4	2	$\mathbb{R}^2$
2	1	$\mathbb{R}$
0	1	$\mathbb{R}$