**Project 3: Numerical Integration of Stochastic Differential Equations**

1. Write down an algorithm to simulate a Wiener process with mean zero and unit variance.

2. Show that for a Wiener process \( W_t \) we have
   \[
   W_t dW_t = \frac{1}{2} d(W_t^2) - \frac{1}{2} dt.
   \]

3. Consider the following SDE
   \[
   dX_t = \mu X_t dt + \sigma X_t dW_t
   \]
   which describes geometric Brownian motion and which is used in option pricing. The stochasticity can here be thought of as a fluctuating growth rate \( a(t) \) around a mean growth rate \( \mu \) in the growth equation \( dX/dt = a(t)X \).
   Integrate this equation using (a) the Euler-Maruyama scheme and (b) the Milstein scheme. Generate equidistant approximations on the time interval \([0, 1]\) with \( X_0 = 1.0, \mu = 1.5 \) and \( \sigma = 1.0 \). Verify numerically that the Euler-Maruyama scheme has strong convergence \( O(\Delta t^{1/2}) \) whereas the Milstein scheme has strong convergence \( O(\Delta t) \). Use the exact solution to determine the error of the approximation.
   By creating an ensemble of trajectories determine numerically the order of weak convergence for both schemes.

4. Consider the following stochastic equation
   \[
   dy = (\mu ty - y^3)dt + b dW,
   \]
   where the stochastic term models additive environmental noise. By numerically integrating this model study the influence of the noise term on the dynamics of this nonlinear dynamical system. In particular study the influence of the strength of noise level \( b \) (keep \( \mu = 0.03 \) and vary \( b \)). Use \( y(0) = 0 \) as an initial condition. Interpret your result - use what you know about the deterministic case \( b = 0 \).