The Zakharov-Kuznetsov Equation as a Two-Dimensional Model for Nonlinear Rossby Waves

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Abstract

We derive the Zakharov-Kuznetsov equation for large scale motion from the barotropic quasigeostrophic equation in a weakly nonlinear, long wave approximation. We study the evolution of finite amplitude perturbations of an unstable jet in planetary atmospheres and oceans. We use multiple scale analysis combined with asymptotic matching. The Zakharov-Kuznetsov equation is suggestive that the perturbations may develop into coherent vortex-like structures.
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1 Introduction

Planetary atmospheres and oceans are strongly turbulent media. However, highly ordered coherent structures arise in a process of self-organization, and dominate the dynamics on slow temporal and large spatial scales.

The spontaneous appearance of coherent structures is a characteristic of two-dimensional fluid flows. The basic underlying structure of these flows is linked to the existence of two quadratic, positive definite invariants, energy and enstrophy. In spectra of two-dimensional turbulence one observes two different cascades associated with these conserved quantities; a direct enstrophy cascade towards small spatial scales, and an indirect energy cascade towards larger spatial scales. It is the latter which gives rise to vortex merging leading to larger and larger vortices. In this paper we view the problem as one of weakly nonlinear hydrodynamic stability rather than turbulence phenomenology.

Most vortices are monopolar, but dipole and even tripole vortices can also appear spontaneously. Monopole vortices are mainly created by shear flow instabilities whereas dipole vortices typically appear when some additional forcing is applied to the flow.

The richness and complexity of two-dimensional flows and the simultaneous presence of motion on very different temporal and spatial scales makes a direct analysis of the basic equations of motion very difficult. The presence of rotation reinforces the two-dimensional character in accordance with the Taylor-Proudman theorem, but rotation can also introduce baroclinic instability. The latter is a three-dimensional feature and, thus supports a direct energy cascade towards small scales. Hence, the dynamics is determined by competing two-dimensional and three-dimensional processes [22, 1, 2, 24]. To study vortices in geophysical fluid dynamics the primitive equations are further reduced by approximations which allow us to focus on temporal and spatial length scales of vortices [28, 29]. If additionally baroclinic processes are excluded a further simplification can be made. The dynamically important variable is the so called potential vorticity q. The resulting quasigeostrophic barotropic vorticity equation

$$\frac{D}{Dt}q = 0 \quad \text{where} \quad q = \frac{\Delta\psi + f(y)}{H} \tag{1}$$

describes large scale motion on a slow time scale. Here ψ is the stream function, $f(y)$ describes the ambient rotation of the planet and H is the fluid depth. This equation was first derived by Charney [4], and then independently by Obukhov [26]. In the context of low-frequency drift waves in magnetized plasmas Equation (1) is known as the Hasegawa-Mima equation [11]. This equation has been the mathematical starting point for much of the research done on coherent structures and vortices. It supports so called modons which are localized soliton-like coherent solutions. Exact modon solutions were obtained by Larichev & Reznik [18] for a stationary double-vortex solution which is antisymmetric in longitude. Extensions to more general solutions have been made [5, 9], and the spherical geometry of planets has been incorporated [30, 31, 25]. However, modons have the drawback that the potential vorticity is not a smooth function of the stream function, but may be multivalued. Therefore interest has grown in low-dimensional models, although a rigorous proof of existence of a low-dimensional attractor in quasigeostrophic systems is still an unsolved problem. Strictly speaking, one can only define a "slowest invariant manifold" [3], since the small-scale events, i.e. the high-frequency and high-wavenumber processes, enlarge the Hausdorff dimension for the attractor without any convergence [33]. Nevertheless, in order to understand better the particular mechanisms involved in the formation and dynamics of vortices in geophysical fluid dynamics, it is useful to perform asymptotic techniques to derive reduced amplitude equations for the basic quasigeostrophic equations in a multiple scale analysis and study the derived model evolution-equations. The basic idea here is that coherent vortices may be identified with solitary wave solutions of generic nonlinear dispersive wave equations.

Most research has been done in the frame work of the Korteweg-de Vries equation [21, 32, 27, 10, 19, 20, 23, 6, 7] or in the framework of the Boussinesq equation [12, 13]. While these models were helpful in describing and identifying mechanisms for atmospheric blocking, cyclogenesis, meandering of oceanic streams and the persistence of the Great Red Spot in the Jovian atmosphere, they are all one-dimensional models with their obvious limitations.

In this Paper we will extend weakly nonlinear, long wave multiple scale analysis to two dimensions and derive the Zakharov-Kuznetsov equation

$$A_T + \Delta A_X - \mu A A_X - \xi A_{XXX} - \zeta A_{XYY} = 0 \ .$$

The Zakharov-Kuznetsov equation [34] is one of two well-studied canonical two-dimensional extensions of the Korteweg-de Vries equation [17]; the other being the Kadomtsev-Petviashvilli equation [16]. In contrast to the Kadomtsev-Petviashvilli equation, the Zakharov-Kuznetsov equation has so far never been derived in a geophysical fluid dynamics context. For a derivation of the Kadomtsev-Petviashvilli equation for internal waves, see [8]. Whereas the Kadomtsev-Petviashvilli equation is valid in isotropic situations, the Zakharov-Kuznetsov equation is valid in anisotropic settings which is exactly the case for rotating fluids where the differential latitudinal dependence of the rotation rate causes anisotropy between the meridional and the longitudinal directions. Moreover, in contrast to the Kadomtsev-Petviashvilli equation the Zakharov-Kuznetsov equation supports stable lump solitary waves. This makes the Zakharov-Kuznetsov equation a very attractive model equation for the study of vortices in geophsyical flows.

The paper is organized as follows. In Section 2 we set up the barotropic vorticity equation and the mean flow configurations under consideration. In the beginning of Section 3 we will give a simple heuristic scaling argument based on the linearized barotropic vorticity equation to motivate why the Zakharov-Kuznetsov equation is the generic two-dimensional nonlinear wave equation. In the remainder of Section 3 we will derive the Zakharov-Kuznetsov equation in an asymptotic multiple scale analysis. Section 4 concludes the Paper with a discussion and an outlook on further research.

2 Barotropic Quasigeostrophic Equation

Separating the meanflow U from the perturbation pressure fields and using the Boussinesq approximation, we obtain the following equation for the perturbation pressure field [28]

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) q + \psi_x Q_y + J\left(\psi, q\right) = 0 \ , \tag{2}$$

where

$$q = \nabla^2\psi - F\psi \; ,$$
$$Q_y = \beta - U_{yy} + FU \; ,$$

with Froude number F and the Jacobian defined by $J(a,b) = a_x b_y - a_y b_x$. We investigate a mean flow which consists of a constant meanflow U_m and a superimposed narrow shear jet with opposite flow direction confined between $y = \pm L$ (see Fig.1). As we will see, the non-vanishing slope at at least one boundary of the localized jet is important. The boundary conditions are $\psi = \text{const}$ at $y = \pm\infty$, and we require that the narrow shear jet forms a transport barrier to the flow, i.e. we require $\psi = \text{const}$ at $y = \pm L$.

3 Nonlinear Wave Equation

3.1 Linear Dispersion Relation

Before we consider the weakly nonlinear, long wave approximation, it is useful to discuss some properties of the linearised version of equations (2) with a non-constant meanflow in terms of a normal mode analysis [28]. Linearisation of equation (2) yields

$$\left(\frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}\right) q + \frac{\partial \psi}{\partial x}\frac{\partial Q}{\partial y} = 0 \; . \tag{3}$$

In terms of $\psi = \Re\{\Phi(y)\exp(ik\,(x - ct))\}$ one obtains the necessary condition for barotropic instability that $\beta - d^2U/dy^2$ must change sign at least once within the confines of the domain. For a meanflow as depicted in Fig. 1 instability is possible provided that the jet is sufficiently strong in amplitude and sufficiently narrow. It is this unstable region of exponentially growing modes for which a nonlinear model is primarily needed.

In the following we motivate why we expect the weakly nonlinear evolution equation for finite amplitude disturbances of a jet to be of the Zakharov-Kuznetsov type. We motivate our approach by looking at the linearization (3) for constant meanflow $U(y) = \bar{U}$. In terms of $\psi = a_0 \exp i(k(x - ct) + ly)$ we obtain the dispersion relation

$$c = \bar{U} - \frac{Q_y}{k^2 + l^2 + F} \; .$$

If we focus on dynamics on the large spatial and slow temporal time scales $X = \epsilon x, Y = \epsilon y$, for some appropriately chosen ϵ (see Section 3.2.1), i.e. small wavenumbers k, l, and $T = \epsilon^3 t$ this is suggestive of the Zakharov-Kuznetsov equation

$$A_T + \Delta A_X - \mu A A_X - \xi A_{XXX} - \zeta A_{XYY} = 0 \; . \tag{4}$$

The reason why we obtain an equation of the Zakharov-Kuznetsov type instead of the usual Kadomtsev-Petviashvilli type mostly encountered in fluid systems is the anisotropic character of (2) caused by the β-effect.

3.2 Weakly Nonlinear Model

We consider weakly nonlinear waves riding on a background meanflow. The meanflow consists of a constant part U_m and a strong but narrow jetstream with opposite flow direction to the mean flow (see Fig. 1). The narrow jet is located on a short meridional scale y. In the outer region the problem (2) can be reduced to the linear problem (3) with constant meanflow U_m. In the interior the structure of the jet does not allow for sinusoidal wave solutions but instead we will derive a nonlinear wave equation. In order for the nonlinear wave equation which is valid only in the inner region where the jet is nonuniform, the inner solution has to be matched to the outer sinusoidal solution.

3.2.1 Inner solution

In the interior of the jet on the small scale y, the meanflow is not constant. We shall study weakly nonlinear long waves. We therefore scale our variables according to $X' = X/R_d$, $Y' = Y/R_d$, where R_d denotes the Rossby radius of deformation, $y' = y/L$, where L is the width of the jet, $t' = f_0 t$, where f_0 is a typical Coriolis parameter, $U' = \mathcal{U}U$, where $\mathcal{U}$ is a typical mean flow velocity, and $\psi' = \psi/(f_0 R_d^2)$. This allows us to explicitly define the small parameter $\epsilon = L/R_d$, and the following scales (omitting the primes),

$$
\begin{aligned}
&X = \epsilon x, \qquad Y = \epsilon y, \qquad T = \sqrt{\epsilon}\epsilon^3 t, \\
&\psi(X, Y, T, y) = \sqrt{\epsilon}\epsilon^2\psi^{(0)} \;+\; \sqrt{\epsilon}\epsilon^3\psi^{(1)} \;+\; \sqrt{\epsilon}\epsilon^4\psi^{(2)} \;+\; \cdots \,.
\end{aligned}
$$

We require $f_0 R_d/\mathcal{U} = \epsilon^2$ which states that the wave speed of the weakly nonlinear waves are small compared to the mean flow. The mean flow within the narrow jet region is assumed to scale as $U \to \sqrt{\epsilon}U$. Next, we rescale the parameters $F \to \epsilon^2 F$ and $\beta \to \sqrt{\epsilon}\epsilon^2\beta$. The scaling of the Froude numbers implies that our model is valid for situations where the internal Rossby radius of deformation is of the order of the long horizontal scale. Further, the scaling of β implies that $Q_y \approx -U_{yy}$ at the lowest order. This scaling (motivated by our analysis in Section 3.1) assures that we obtain the Zakharov-Kuznetsov equation. The boundary conditions we use are $\psi = \text{const}$ at $y = \pm\infty$ and $\psi = \text{const}$ at $y = \pm L$.
Substituting this scaling into equation (2) yields,

$$
\begin{aligned}
0 \;=\; & (\epsilon^3\partial_T + \epsilon U\partial_X)(\epsilon^4\partial_{XX}\psi^{(0)} + \epsilon^2\partial_{yy}\psi^{(0)} + 2\epsilon^3\partial_{yY}\psi^{(0)} + \epsilon^4\partial_{YY}\psi^{(0)} - \epsilon^4 F\psi^{(0)} \\
& \qquad + \epsilon^3\partial_{yy}\psi^{(1)} + \epsilon^4\partial_{yy}\psi^{(2)} + 2\epsilon^4\partial_{yY}\psi^{(1)}) \\
+ \; & \epsilon^3\psi_X^{(0)}(\epsilon^2\beta - U_{yy} + \epsilon^2 FU) \;+\; \epsilon^4\psi_X^{(1)}(-U_{yy}) \;+\; \epsilon^5\psi_X^{(2)}(-U_{yy}) \\
+ \; & \epsilon^5(\psi_X^{(0)}\psi_{yyy}^{(0)} - \psi_y^{(0)}\psi_{yyX}^{(0)}) \,,
\end{aligned}
\tag{5}
$$

where we ignored a common factor of ϵ. Counting the orders of ϵ we obtain to the lowest order, $\mathcal{O}(\epsilon^3)$,

$$
U\psi_{yyX}^{(0)} - U_{yy}\psi_X^{(0)} = 0 \,,
$$

which can be written as

$$
\mathcal{L}\psi_X^{(0)} = 0 \,, \tag{6}
$$

where

$$
\mathcal{L} = \frac{\partial}{\partial y}\left[U\partial_y - U_y\right] \,. \tag{7}
$$

We look for an amplitude equation, i.e. we want to write

$$
\psi^{(0)}(X, Y, T, y) = A(X, Y, T)\varphi^{(0)}(y) \tag{8}
$$

and seek an evolution equation for the slowly varying amplitude $A(X, Y, T)$. We easily find

$$
\varphi^{(0)}(y) = U(y)\left(1 + \alpha_0\int_{-L}^{y}\frac{1}{U^2(y')}\,dy'\right) \,, \tag{9}
$$

where α_0 is a constant of integration. As will be shown in Section 3.3, this constant is determined by the asymptotic matching. To simplify the upcoming formulae we anticipate the result and set $\alpha_0 = 0$. We summarize the solution of equation (6)

$$
\psi^{(0)}(X, Y, T, y) = A(X, Y, T)U(y) \,. \tag{10}
$$

Hence the meridional structure on the small scale y of ψ is entirely determined by the mean flow at the leading order.

At the next order, $\mathcal{O}(\epsilon^4)$, we obtain a linear inhomogeneous equation for $\psi^{(1)}$,

$$2U\psi^{(0)}_{yYX} + U\psi^{(1)}_{yyX} - U_{yy}\psi^{(1)}_X = 0 \ ,$$

which can be written, using (9), as

$$\mathcal{L}\psi^{(1)}_X = -2U\varphi^{(0)}_y A_{XY} \ .$$

This equation is again solved by the method of variation of parameters and we obtain

$$\psi^{(1)} = \varphi^{(1)} A_Y \ , \tag{11}$$

with

$$\varphi^{(1)} = U(y)\left(1 - y - L + \alpha_1 \int_{-L}^{y} \frac{1}{U^2(y')} dy'\right) \ .$$

We note that the higher order term $\psi^{(1)}$ is slaved to the $\psi^{(0)}$ term and the dynamics of the corresponding amplitude equation for A which will be derived shortly. To assure the existence of the integral over $U^2(y)$ we need to impose restrictions on the narrow jet stream and its behaviour at the boundaries $\pm L$ where $U(\pm L) = 0$. In order for the integral over $U^2(y)$ to be defined near the boundaries where $U(\pm L) = 0$, we have to restrict the mean flow configurations such that $1/U(y)$ has a weaker slope than $1/\sqrt{L^2 - y^2}$ near the boundaries at $y = \pm L$.

Note that due to the structure for the stream function $\psi = U(y)g(A, \epsilon)$ where the function g is defined through (10) and (11), the boundary condition $\psi = \text{const}$ at $y = \pm L$ does not restrict the dynamics of the amplitude $A(X, Y, T)$. The boundary condition is exactly satisfied via $U(\pm L) = 0$.

The $\mathcal{O}(\epsilon^5)$-terms provide us with an evolution equation for the amplitude A. We obtain

$$\begin{aligned} & U\psi^{(2)}_{Xyy} - U_{yy}\psi^{(2)}_X + 2U\psi^{(1)}_{yYX} \\ + \ & \{\partial_T \psi^{(0)}_{yy} + U\psi^{(0)}_{XXX} + U\psi^{(0)}_{XYY} - FU\psi^{(0)}_X \\ + \ & \psi^{(0)}_X \psi^{(0)}_{yyy} - \psi^{(0)}_y \psi^{(0)}_{Xyy} + \beta\psi^{(0)}_X + FU\psi^{(0)}_X\} = 0 \ , \end{aligned}$$

which, using (8), can be written as

$$\mathcal{L}\psi^{(2)}_X = -G \ , \tag{12}$$

where

$$\begin{aligned} G \ = \ & \varphi^{(0)}_{yy} A_T + U\varphi^{(0)}(A_{XXX} + A_{XYY}) \\ + \ & \varphi^{(0)}\beta A_X + \left(\varphi^{(0)}\varphi^{(0)}_{yyy} - \varphi^{(0)}_y \varphi^{(0)}_{yy}\right) A A_X \\ + \ & 2U\varphi^{(1)}_y A_{XYY} \ . \end{aligned} \tag{13}$$

To assure boundedness of the solutions of (12) we have to require a solvability condition in form of a Fredholm alternative. For that purpose we need to determine the eigenfunctions of the adjoint problem using the boundary conditions $\psi = \text{const}$ at $y = \pm L$.
The homogeneous free adjoint problem to equation (12) may be written as

$$\mathcal{L}^{\dagger}\phi = 0 \tag{14}$$

with

$$\mathcal{L}^\dagger = 2U_y\partial_y + U\partial_{yy} \ .$$

The adjoint eigenvalue problem (14) has one trivial constant kernel mode $\phi_1 = const$ and one nontrivial, namely

$$\phi_2(y) = \int_0^y \frac{1}{U^2(y')} dy' \ . \tag{15}$$

The nontrivial kernel mode ϕ_2 has to be discarded because it does not satisfy the boundary condition of the dual problem. The corresponding dual boundary condition is $\partial_y\phi = 0$ at $y = \pm L$ which is only satisfied by the constant kernel mode ϕ_1. Note that if the operator $\mathcal{L}$ were elliptic (i.e. U strictly positive or negative) in the interval $[-L, L]$ as is the case for the mean flow configuration depicted in Fig. 1, there would exist a unique (up to a scalar) positive eigenfunction which would be ϕ_1 in this case.
The solvability condition is thus given by the trivial constant kernel mode and reads as

$$\int_{-L}^{L} G \ dy = 0 \ . \tag{16}$$

On substituting the solution of the lowest orders (10) and (11) into (16) we obtain the desired amplitude equation for A,

$$A_T + \Delta A_X - \mu A A_X - \xi A_{XXX} - \zeta A_{XYY} = 0 \ , \tag{17}$$

where

$$\begin{aligned} I &= -\left[U_y\right]_{-L}^{L} \ , \\ I\Delta &= -\int_{-L}^{L} \beta U \ dy \ , \\ I\mu &= -\left[U_y^2\right]_{-L}^{L} \ , \\ I\xi &= \int_{-L}^{L} U^2 \ dy \ , \\ I\zeta &= 2L\alpha_1 \ . \end{aligned} \tag{18}$$

Note that our scaling $U \to \sqrt{\epsilon}U$ implies that the coefficients μ and ξ are of $\mathcal{O}(\sqrt{\epsilon})$. We note that due to the last term of (13) the Zakharov-Kuznetsov equation is inhomogeneous in the sense that $\xi \neq \zeta$. Equation (17) can be cast in the standard homogeneous form by formally rescaling X and Y. The coefficients of the nonlinear terms μ require a non-vanishing slope of the mean flow at at least one boundary. The so far free constant of integration α_1 will be determined in Section 3.3 via asymptotic matching conditions.

3.2.2 Outer solution

In the outer region where the meanflow is uniform and constant, (2) reduces to the simple linear equation (3) for the streamfunction with constant coefficients [28]. The governing equation in the outer region using our scaling can be obtained from (5) by setting $\partial_y = 0$, and is

$$U_m(\psi_{XX} + \psi_{YY} - F\psi)_X + Q_y\psi_X = 0 \ ,$$

where $Q_y = \beta + FU_m$. The solution of the streamfunction can be expressed as a Fourier transform

$$\psi^{(\text{out})} = \int_{-\infty}^{\infty} a(T,k)\exp(ikX + ilY) \ dk \tag{19}$$

where l is the meridional wave number and is constrained by the dispersion relation of the linearized model (3) via $l^2 + k^2 = Q_y/U_m - F$. To assure that information travels outward of the jet region we require $\text{sign}(Yl) > 0$ at the boundary as a causality condition.

3.3 Asymptotic Matching

At the relevant orders the inner solution (10) and (11), and the outer solution (19) have to be matched at some intermediate scale. The outer solution has been derived on the large scale Y whereas the inner solution and its associated amplitude equation, the Zakharov-Kuznetsov equation (17), were derived on the short scale y. In the process of multiscale analysis we treated y and Y as independent variables. This allowed the inner solution (10) and (11) to depend on the outer variable Y. Asymptotic matching is performed by expressing the inner solution in terms of the outer variable Y, and the outer solution in terms of the inner variable y. We then require that the two solutions match up in the asymptotic limit $\epsilon \to 0$. In this procedure, we do not consider the long and the short scales, Y and y as independent anymore, but to be related via $Y = \epsilon y$, determined by our scaling.

We recall the expression for the inner solution $\psi^{(\mathrm{in})}$

$$\begin{aligned} \psi^{(\mathrm{in})}(X,y,Y,T) &= U(y)\left(1+\alpha_0\int_{-L}^{y}\frac{1}{U^2(y')}\,dy'\right)A(X,Y,T) \\ &+ \epsilon U(y)\left(1-y-L+\alpha_1\int_{-L}^{y}\frac{1}{U^2(y')}dy'\right)A_Y(X,Y,T)\ . \end{aligned} \tag{20}$$

To restrict this solution to the inner region $y \in [-L, L]$, we Taylor-expand the solution around $Y = 0$, and obtain

$$\begin{aligned} \psi^{(\mathrm{in})}(X,y,T) &= U(y)\left(1+\alpha_0\int_{-L}^{y}\frac{1}{U^2(y')}\,dy'\right)A(X,0,T) \\ &+ \epsilon U(y)\left(1-L+(\alpha_0+\alpha_1)\int_{-L}^{y}\frac{1}{U^2(y')}dy'\right)A_Y(X,0,T)+\mathcal{O}(\epsilon^2)\ . \end{aligned} \tag{21}$$

This solution is valid in the inner region and does not contain any outer variables Y.
The asymptotic matching is performed by expressing the inner solution (21) in terms of the outer variable Y, and then taking the limit $\epsilon \to 0$. The inner solution becomes up to order $\mathcal{O}(\epsilon)$

$$\begin{aligned} \psi^{(\mathrm{in})}(X,Y,T;\epsilon) &= U_m\left(1+\alpha_0\lim_{\epsilon\to 0}\int_{-L}^{Y/\epsilon}\frac{1}{U^2(y')}\,dy'\right)A(X,0,T) \\ &+ \epsilon U_m\left(1-L+(\alpha_0+\alpha_1)\lim_{\epsilon\to 0}\int_{-L}^{Y/\epsilon}\frac{1}{U^2(y')}dy'\right)A_Y(X,0,T)\ . \end{aligned} \tag{22}$$

The solution (22) has now to be matched with the corresponding outer solution. We recall the expression for the outer solution $\psi^{(\mathrm{out})}$

$$\psi^{(\mathrm{out})}(X,Y,T) = \int_{-\infty}^{\infty} a(T,k)\exp(ikX+ilY)\ dk\ , \tag{23}$$

where the wave numbers k and l are related through the linear dispersion relation. The outer solution is expressed entirely in terms of the outer large variable Y. We expand the outer solution in orders of ϵ by expanding the Fourier amplitude $a(T,k)$ in a series in ϵ according to $a(T,k) = a_0(T,k) + \epsilon a_1(T,k)$. Analogously to the procedure above for the inner solution, we express now the outer solution in terms of the inner variable y, and take the limit $\epsilon \to 0$. We obtain

$$\begin{aligned} \psi^{(\mathrm{out})}(X,y,T;\epsilon) &= \int_{-\infty}^{\infty} a_0(T,k)\exp(ikX)\ dk \\ &+ \epsilon\left(\int_{-\infty}^{\infty} a_1(T,k)\exp(ikX)\ dk + y\frac{\partial}{\partial Y}\int_{-\infty}^{\infty} a_0(T,k)\exp(ikX)\ dk\right)+\mathcal{O}(\epsilon^2) \end{aligned} \tag{24}$$

The two solutions (22) and (24) have to match up on an intermediate scale. We need to require that the asymptotic limit of the outer solutions (24) coincides with the asymptotic limit of the inner solution (22). The asymptotic matching condition at the lowest order, $\mathcal{O}(1)$, is

$$U_m \left(1 + \alpha_0 \lim_{\epsilon\to 0} \int_{-L}^{Y/\epsilon} \frac{1}{U^2(y')}\, dy'\right) A(X,0,T) = \int_{-\infty}^{\infty} a_0(k,T) \exp(ikX)\, dk \; . \tag{25}$$

In order to assure boundedness of the solutions we need to impose $\alpha_0 = 0$. We note that we used this result in the previous Sections to derive all higher order expressions. We obtain a relationship between the Fourier amplitude $a_0(T,k)$ and the solution of the Zakharov-Kuznetsov equation $A(X,Y,T)$

$$\int_{-\infty}^{\infty} a_0(k,T) \exp(ikX)\, dk = U_m A(X,0,T) \; . \tag{26}$$

The asymptotic matching condition at $\mathcal{O}(\epsilon)$ reads as

$$U_m \left(1 - L + \alpha_1 \lim_{\epsilon\to 0} \int_{-L}^{Y/\epsilon} \frac{1}{U^2(y')}\, dy'\right) A_Y(X,0,T) = U_m A_Y(X,0,T) \frac{Y}{\epsilon} + \int_{-\infty}^{\infty} a_1(k,T) \exp(ikX)\, dk \; , \tag{27}$$

where we used (26).
The integral on the left-hand side of equation (27) which involves the remaining free constant of integration α_1 can be expanded as

$$\alpha_1 \left(\int_{-L}^{L} \frac{1}{U^2(y')}\, dy' + \lim_{\epsilon\to 0} \int_{L}^{Y/\epsilon} \frac{1}{U^2(y')}\, dy' \right) = \alpha_1 \left(\int_{-L}^{L} \frac{1}{U^2(y')}\, dy' + \frac{1}{U_m^2} (\lim_{\epsilon\to 0} \frac{Y}{\epsilon} - L) \right) + \mathcal{O}(\epsilon) \; .$$

Hence α_1 is determined by balancing and bounding the terms proportional to Y in (27), and we obtain $\alpha_1 = U_m^2$.
The asymptotic matching slaves the first-order Fourier amplitude correction $a_1(T,k)$ to the dynamics of the Zakharov-Kuznetsov equation, and we have

$$\int_{-\infty}^{\infty} a_1(k,T) \exp(ikX)\, dk = U_m \left(1 - 2L + U_m^2 \int_{-L}^{L} \frac{1}{U^2(y')}\, dy'\right) A_Y(X,0,T) \; . \tag{28}$$

The conditions (26) and (28) extend the dynamics of the Zakharov-Kuznetsov equation to the outer region via a Fourier transformation, and relate the outer Fourier amplitude $a(T,k)$ to the solution of the Zakharov-Kuznetsov equation $A(X,Y,T)$.
The asymptotic matching has fixed the two constants of motion, α_0 and α_1, and removed any arbitrariness in the parameters of the Zakharov-Kuznetsov equation (18). The meridional dispersive parameter has now been found to be $\zeta = 2LU_m^2$.

4 Discussion

We have derived the nonlinear dispersive Zakharov-Kuznetsov equation from the quasigeostrophic barotropic vorticity equation. It is well known that the Zakharov-Kuznetsov equation, although it is not integrable by means of the inverse scattering transform, supports a family of steady-shape stable lump solitary waves, moving at an arbitrary velocity [34, 15]. These may help to describe two-dimensional coherent structures such as atmospheric blocking events, long lived eddies in the ocean or coherent structures in the Jovian atmosphere such as the Great Red Spot [14]. The model is from an analytical point of view easier to treat than the full barotropic quasigeostrophic equation and its solutions do not exhibit multivalued potential vorticity-stream function relationships as modons do.

We have assumed a meanflow U which consists of a constant part U_m and a narrow localized jet with opposite flow direction. Such persistent shear layers exist between the zones and belts in the Jovian atmosphere. In the derivation of the Zakharov-Kuznetsov equation (17) we imposed a restriction onto the y-dependence of the mean flow at the boundaries $y = \pm L$. We require the y-dependence to be as least of square-root behaviour. If we assume such a square root behaviour the maximal value of U in the interior of the narrow jet region is restricted to be of order $\sqrt{\epsilon}$ (unless we allow for artificial additional inflection points within the narrow jet region). This justifies our initial scaling $U \to \sqrt{\epsilon}U$.
The coefficient ζ defined in (18) is responsible for the two-dimensionality of the Zakharov-Kuznetsov and is the non-trivial extension to the Korteweg-de Vries type equations which had been derived in this context so far [21, 32, 27, 10, 19, 20, 23, 6, 7]. In order that the meridional dispersive ζ is of the same order as the longitudinal dispersive coefficient ξ, we need to require that the magnitude of the asymptotic mean flow U_m is of order $\mathcal{O}(1)$.

To test whether the model equation (17) really is an adequate model, the stability of its solutions has to be numerically tested within the Zakharov-Kuznetsov system and also within the barotropic quasi-geostrophic equation. The Zakharov-Kuznetsov equation has been derived using asymptotic techniques and is as such an asymptotic limit to the barotropic quasigeostrophic vorticity equation. However, it is not clear that the same is true for the solutions. The solutions of the Zakharov-Kuznetsov equation do not necessarily have to be asymptotically close to the solutions of the full quasigeostrophic system. This is due to the lack of a centre manifold of the quasigeostrophic system as discussed in the introduction. In further work it will be interesting to test the approximation of the solution numerically by taking solutions of the Zakharov-Kuznetsov equation and testing their dynamics in the full quasigeostrophic system.

We will briefly discuss the structure of the Zakharov-Kuznetsov equation. Geophysical flow on large scales is widely accepted to be conservative. This allows for Hamiltonian descriptions of the flow on large scales. Our model also exhibits a Hamiltonian structure. Note that the momentum

$$P = \int_{-\infty}^{\infty} A^2 \, dX \, dY$$

and the Hamiltonian with the Hamiltonian density

$$\mathcal{H} = \frac{\xi}{2} A_X^2 + \frac{\zeta}{2} A_Y^2 - \mu A^3$$

are conserved.

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Figures

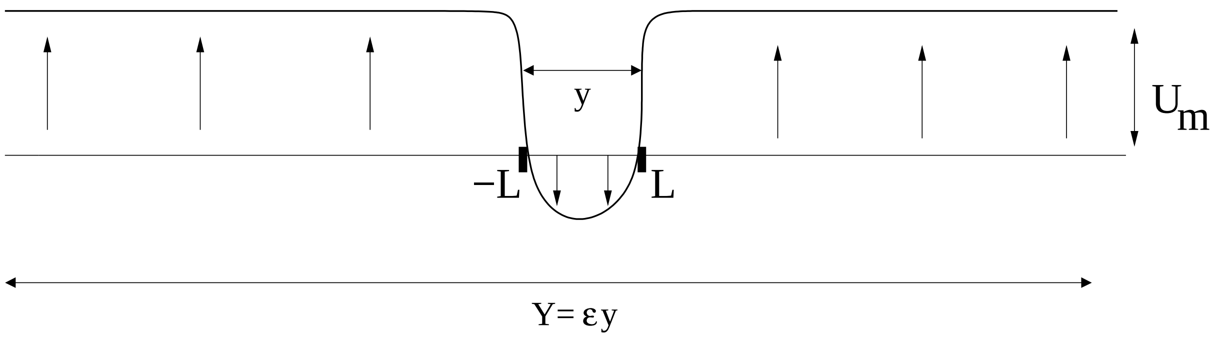

Figure 1: Sketch of a typical mean flow.