Kazhdan-Lusztig cells in affine Weyl groups (with unequal parameters)

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Let $V$ be an euclidean space of dim $r$, with inner product $\langle ., . \rangle$. A root system $\Phi$ is a set of non-zero vectors that satisfy the following:

- $\Phi$ spans $V$.
- For $\alpha \in \Phi$, we have $\mathbb{R} \alpha \cap \Phi = \{ \alpha, -\alpha \}$.
- For $\alpha \in \Phi$, let $\sigma_\alpha$ the orthogonal reflection with fixed point set the hyperplane perpendicular to $\alpha$. We have $\sigma_\alpha(\Phi) = \Phi$.
- For any $\alpha, \beta \in \phi$ we have $\frac{2 \langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.

Let $\Phi$ be an “irreducible” root system. The Weyl group of $\Phi$ is the group generated by $\{ \sigma_\alpha | \alpha \in \Phi \}$. It has a presentation of the form:

$$\{ \sigma_\alpha_1, ..., \sigma_\alpha_r | (\sigma_\alpha_i \sigma_\alpha_j)^{m_{i,j}} = 1, \sigma_{\alpha_i}^2 = 1 \}$$

where $m_{i,j} \in \{2, 3, 4, 6\}$ for $i \neq j$. 
Let $Q$ be the lattice generated by $\Phi$:

$$\{n_1\alpha_1 + \ldots + n_k\alpha_k \mid n_i \in \mathbb{Z}, \alpha_i \in \Phi\}$$

The Weyl group $W_0$ of $\Phi$ acts on $Q$.

Thus we can form the semi-direct product:

$$W := W_0 \rtimes Q$$

This is the affine Weyl group associated to $\Phi$. 
$V$: Euclidean space of dimension $r$.

$\Phi$: Irreducible root system of $V$.

For any $\alpha \in \Phi$ and $k \in \mathbb{Z}$ let:

$$H_{\alpha,k} = \{ x \in V \mid \frac{2\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} = k \}$$

The Weyl group $W_0$ of $\Phi$ is generated by the orthogonal reflections with fixed point set $H_{\alpha,0}$. We have:

$$W_0 = \langle \sigma_1, \sigma_2 \mid (\sigma_1 \sigma_2)^6 = 1, \sigma_i^2 = 1 \rangle$$
$V$: Euclidean space of dimension $r$.

$\Phi$: Irreducible root system of $V$.

For any $\alpha \in \Phi$ and $k \in \mathbb{Z}$ let:

$$H_{\alpha,k} = \{ x \in V \mid \frac{2\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} = k \}$$

The Weyl group $W_0$ of $\Phi$ is generated by all the orthogonal reflections with fixed point set $H_{\alpha,0}$. We have:

$$W_0 = \langle \sigma_1, \sigma_2 \mid (\sigma_1 \sigma_2)^6 = 1, \sigma_i^2 = 1 \rangle$$

The affine Weyl group $W$ of $\Phi$ is generated by all the orthogonal reflections $\sigma_{H_{\alpha,k}}$ with fixed point set $H_{\alpha,k}$.

Here we have:

$$W = \langle \sigma_1, \sigma_2, \sigma_3 \mid (\sigma_1 \sigma_2)^6 = 1, (\sigma_2 \sigma_3)^3 = 1, (\sigma_1 \sigma_3)^2 = 1, \sigma_i^2 = 1 \rangle$$
Affine Weyl groups  

Classification

\[ \tilde{A}_1: \bullet_{\infty} \]

\[ \tilde{B}_2 = \tilde{C}_2: \bullet \quad \bullet \quad \bullet \]

\[ \tilde{C}_n: \bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet \]

\[ \tilde{E}_7: \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

\[ \tilde{E}_8: \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

\[ \tilde{E}_6: \bullet \quad \bullet \quad \bullet \quad \bullet \]

\[ \tilde{A}_n: \bullet \quad \bullet \quad \cdots \quad \bullet \]

\[ \tilde{B}_n: \bullet \quad \bullet \quad \cdots \quad \bullet \]

\[ \tilde{C}_n: \bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet \]

\[ \tilde{D}_n: \bullet \quad \bullet \quad \cdots \quad \bullet \]

\[ \tilde{E}_7: \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

\[ \tilde{E}_8: \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

\[ \tilde{G}_2: \bullet \quad \bullet \]

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Let $W$ be an affine Weyl group with generating set $S$.

For $w \in W$, let $\ell(w)$ be the smallest integer $n \in \mathbb{N}$ such that $w = s_1 \ldots s_n$ with $s_i \in S$. The function $\ell$ is called the length function.

Let $L$ be a weight function, that is a function $L : W \rightarrow \mathbb{N}$ such that:

\[
L(ww') = L(w) + L(w') \quad \text{whenever } \ell(ww') = \ell(w) + \ell(w')
\]
\[
L(w) > 0 \quad \text{unless } w = 1
\]

The case $L = \ell$ is known as the equal parameter case.

From the above relations, one can see that:

- A weight function $L$ is completely determined by its values on $S$
- Let $s, t \in S$, if the order of $(st)$ is odd, then we must have $L(s) = L(t)$.
\[ \hat{A}_1: \bullet \infty \bullet \]

\[ \hat{B}_2 = \hat{C}_2: \bullet \bullet \bullet \]

\[ \hat{F}_4: \bullet \bullet \bullet \bullet \]

\[ \hat{G}_2: \bullet \bullet \bullet \]

\[ \hat{B}_n: \bullet \bullet \bullet \bullet \bullet \]

\[ \hat{C}_n: \bullet \bullet \bullet \bullet \bullet \]
\[ \tilde{A}_1: \bullet \quad \infty \quad \bullet \]

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\[ \tilde{F}_4: \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

\[ \tilde{B}_n: \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

\[ \tilde{C}_n: \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

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\[ \tilde{A}_1: \quad \infty \]
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Kazhdan–Lusztig theory

Weight functions

\[ \tilde{A}_1 : \bullet_{\infty} \]
\[ \tilde{B}_2 = \tilde{C}_2 : \bullet \bullet \bullet \]
\[ \tilde{B}_n : \bullet \bullet \bullet \bullet \]
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Kazhdan-Lusztig theory  Weight functions

\[ \tilde{A}_1: \quad \begin{array}{c} \infty \end{array} \]

\[ \tilde{B}_2 = \tilde{C}_2: \quad \begin{array}{c} \infty \end{array} \]

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\[ \tilde{C}_n: \quad \begin{array}{c} \infty \end{array} \]

\[ \tilde{G}_2: \quad \begin{array}{c} \infty \end{array} \]
Let \((W, S)\) be an affine Weyl group and \(L\) a weight function on \(W\). Let \(\mathcal{H}\) be the associated Iwahori-Hecke algebra over \(\mathcal{A} = \mathbb{Z}[\nu, \nu^{-1}]\). Standard basis \(\{T_w \mid w \in W\}\) with multiplication

\[
T_s T_w = \begin{cases} 
T_{sw} & \text{if } \ell(sw) > \ell(w) \\
T_{sw} + (\nu^{L(s)} - \nu^{-L(s)})T_w & \text{if } \ell(sw) < \ell(w)
\end{cases}
\]

One can see that \(T_s^{-1} = T_s - (\nu^{L(s)} - \nu^{-L(s)})T_1\).

There is a unique ring involution \(\mathcal{A} \to \mathcal{A}, a \mapsto \bar{a}\), such that \(\bar{\nu} = \nu^{-1}\). We can extend it to a ring involution \(\mathcal{H} \to \mathcal{H}, h \mapsto \bar{h}\), such that:

\[
\sum_{w \in W} a_w T_w = \sum_{w \in W} \bar{a}_w T_{w^{-1}}^{-1} \quad (a_w \in \mathcal{A}).
\]

For any $w \in W$, there exists a unique $C_w \in \mathcal{H}$ such that:

- $\overline{C_w} = C_w$
- $C_w = T_w + \sum_{\ell(y) < \ell(w)} P_{y,w} T_y$ where $P_{y,w} \in v^{-1} \mathbb{Z}[v^{-1}]$

Furthermore, the $C_w$’s form a basis of $\mathcal{H}$ known as the Kazhdan-Lusztig basis.

For example, we have:

$$C_1 = T_1 \quad \text{and} \quad C_s = T_s + v^{-L(s)} T_1$$
Pre-order relation $\leq_L$ defined by:

$$\mathcal{H}C_w \subset \sum_{y \leq_L w} \mathcal{A}C_y$$

Let $s \in S$ and $w \in W$ such that $\ell(w) < \ell(sw)$, then:

$$C_s C_w = C_{sw} + \ldots$$

so we have $sw \leq_L w$

Corresponding equivalence relation $\sim_L$.

The equivalence classes are called left cells.

Similarly we define $\leq_R$, $\sim_R$ and right cells.
We say that $y \leq_{LR} w$ if there exists a sequence:

$$y = y_0, y_1, \ldots, y_n = w$$

such that for any $0 \leq i \leq n - 1$ we have:

$$y_i \leq_L y_{i+1} \text{ or } y_i \leq_R y_{i+1}$$

We get the equivalence relation $\sim_{LR}$ and the two-sided cells.
Structure constants: Write

\[ C_x C_y = \sum_{z \in W} h_{x,y,z} C_z \text{ where } h_{x,y,z} \in A. \]

**G. Lusztig (1985):** Define function \( a: W \rightarrow \mathbb{N}_0 \) by

\[ a(z) = \min\{i \geq 0 \mid v^{-i} h_{x,y,z} \in \mathbb{Z}[v^{-1}] \ \forall x, y \in W\}. \]

If \( W \) is finite, then this function is clearly well defined. In the affine case, it is not clear that this minimum exists! But, it does... Let \( \tilde{v} = L(w_0) \) where \( w_0 \) is the longest element of the Weyl group \( W_0 \) associated to \( W \). We have:

\[ v^{-\tilde{v}} h_{x,y,z} \in \mathbb{Z}[v^{-1}] \text{ for all } x, y, z \in W. \]

In other words, \( a(z) \leq \tilde{v} \) for all \( z \in W \).
The pre-order $\leq_{LR}$ induces a partial order on the two-sided cells.

**Theorem.**

Let

$$c_0 = \{ w \in W \mid a(w) = \tilde{v} \}.$$  

Then $c_0$ is a two-sided cell. Moreover, $c_0$ is the lowest two-sided cell.

Why lowest? Lusztig conjectures:

if $z \leq_{LR} z'$ then $a(z') \leq a(z)$.

Let $z' \in c_0$. Let $z \leq_{LR} z'$. We have:

$$\tilde{v} = a(z') \leq a(z) \leq \tilde{v}$$

which implies $a(z) = \tilde{v}$ and $z \in c_0$. 
• Shi (∼ 1987): $c_0$ is a two-sided cell (equal parameter case).
• Shi (∼ 1988): $c_0$ contains $|W_0|$ left cells (equal parameter).
• Bremke and Xi (∼ 1996): $c_0$ is a two-sided cell (unequal parameter).
• Bremke (∼ 1996): $c_0$ contains at most $|W_0|$ left cells.
• Bremke (∼ 1996): $c_0$ contains $|W_0|$ left cells when the parameters are coming from a graph automorphism

When we know the exact number of left cells in $c_0$, it involves some deep properties of Kazhdan-Lusztig polynomials, such as positivity of the coefficient. Problem: Not true in general!
Example: $G_2$

$V$: Euclidean space of dimension $r$.

$\Phi$: Irreducible root system of $V$.

For any $\alpha \in \Phi$ and $n \in \mathbb{Z}$ let:

$$H_{\alpha,k} = \{x \in V \mid \frac{2\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} = k\}$$

An alcove is a connected component of:

$$V - \bigcup H_{\alpha,k}$$

Denote by $X$ the set of alcoves.

Let $\Omega = \langle \sigma_{H_{\alpha,k}}, k \in \mathbb{Z}, \alpha \in \Phi \rangle$

$\Omega$ acts simply transitively on $X$.

Let $A_0$ be the fundamental alcove:

$$A_0 = \{x \in V \mid 0 < \frac{2\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} < 1\}$$

for all $\alpha \in \Phi^+$. 
A face is a co-dimension 1 facet of an alcove.

Examples: The faces of $A_0$. 
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Examples: The faces of $A_0$.

We look at the orbits of the faces under $\Omega$.

Let $S$ be the set of orbits.

Here we have 3 orbits, namely:

- $s_1 = \text{green}$
- $s_2 = \text{red}$
- $s_3 = \text{blue}$
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Examples: The faces of $A_0$.

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For $s \in S$, we define an involution $A \mapsto sA$ of $X$, where $sA$ is the unique alcove which shares with $A$ a face of type $s$. The set of such maps is a group of permutation of $X$ which is a Coxeter group $W$. We have $W \simeq \Omega$. 
The action of $\mathcal{W}$ on $X$ commutes with the action of $\Omega$. We identify $w \in \mathcal{W}$ with the alcove $wA_0$.

Example:

- alcove $s_3s_2s_1s_2s_3A_0$. 
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- $s_3 A_0$, $s_2 s_3 A_0$, $s_1 s_2 s_3 A_0$,
- $s_2 s_1 s_2 s_3 A_0$, 
- $s_3 s_2 s_1 s_2 s_3 A_0$. 

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- We have $(s_2 s_1)^6 = e$. 
The action of $\mathcal{W}$ on $X$ commutes with the action of $\Omega$.

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- Alcove $s_3 s_2 s_1 s_2 s_3 A_0$.
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- $s_2 s_1 s_2 s_3 A_0$, $s_3 s_2 s_1 s_2 s_3 A_0$,
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- We have $(s_2 s_3)^3 = e$.
The action of $W$ on $X$ commutes with the action of $\Omega$. We identify $w \in W$ with the alcove $wA_0$.

Example:

- alcove $s_3 s_2 s_1 s_2 s_3 A_0$.
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- alcove $s_3 s_2 s_1 s_2 s_3 A_0$.
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- We have $(s_2 s_1)^6 = e$.
- We have $(s_2 s_3)^3 = e$.
- We have $(s_1 s_3)^2 = e$.

Let $s, t \in S$. If a hyperplane $H$ supports a face of type $s$ and a face of type $t$ then $s$ and $t$ are conjugate in $W$. Therefore we can associate to any hyperplane $H$ a weight $c_H = L(s)$ if $H$ supports a face of type $s$. 
Let $w \in W$, we have $\ell(w) =$ number hyperplane which separate $A_0$ and $wA_0$.

Let $x, y \in W$. Consider $yA_0$ and $xyA_0$.
Let \( w \in \mathcal{W} \), we have \( \ell(w) \) = number hyperplane which separate \( A_0 \) and \( wA_0 \).

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First consider the hyperplanes which separate \( A_0 \) and \( yA_0 \);
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First consider the hyperplanes which separate $A_0$ and $yA_0$;
next, the hyperplanes which separate $yA_0$ and $xyA_0$;
finally, let $H_{x,y}$ be the intersection.
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First consider the hyperplanes which separate $A_0$ and $yA_0$; next, the hyperplanes which separate $yA_0$ and $xyA_0$; finally, let $H_{x,y}$ be the intersection.

Let $c_{x,y}$ be...

On this example, we have $c_{x,y} = L(s_2) + L(s_1)$. 

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Let $c_{x,y}$ be...
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**Proposition. G. (~ 2006)**

We have:

$$T_x T_y = \sum_{z \in W} f_{x,y,z} T_z \text{ where } \text{deg}(f_{x,y,z}) \leq c_{x,y}.$$

Let $W' \subseteq W$ be a standard parabolic subgroup, and let $X'$ be the set of all $w \in W$ such that $w$ has minimal length in the coset $wW'$. Let $C$ be a left cell of $W'$. Then $X'.C$ is a union of left cells.
**Theorem.** GECK (~ 2003)

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Let’s take the example of $\tilde{G}_2$ with $W' = \langle s_1, s_2 \rangle$ and parameters:

\[
\begin{align*}
    a & \succ b \\
    s_1 & \quad s_2 & s_3
\end{align*}
\]
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Let’s take the example of $\tilde{G}_2$ with $W' = \langle s_1, s_2 \rangle$ and parameters:

$\begin{align*}
    a & > b \\
    s_1 & = s_2 \\
    s_2 & = s_3 \\
    b & = b
\end{align*}$

Now, $X'A_0$ has the following shape.
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Let $W' \subseteq W$ be a standard parabolic subgroup, and let $X'$ be the set of all $w \in W$ such that $w$ has minimal length in the coset $wW'$. Let $C$ be a left cell of $W'$. Then $X'.C$ is a union of left cells.

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$$a > b \quad s_1 \quad s_2 \quad b \quad s_3$$

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Theorem. \textbf{Geck} (\sim 2003)

Let $W' \subseteq W$ be a standard parabolic subgroup, and let $X'$ be the set of all $w \in W$ such that $w$ has minimal length in the coset $wW'$. Let $\mathcal{C}$ be a left cell of $W'$. Then $X'.\mathcal{C}$ is a union of left cells.

Let's take the example of $G_2$ with $W' = \langle s_1, s_2 \rangle$ and parameters:

Now, $X'A_0$ has the following shape. The decomposition into left cells is as follows. Thus the theorem gives:
\[ W = \tilde{G}_2 : \begin{array}{c} \bullet \\ s_1 \\ \end{array} \begin{array}{c} a \\ b \\ \end{array} \begin{array}{c} \bullet \\ s_2 \\ \end{array} \begin{array}{c} b \\ s_3 \\ \end{array} \]

\[ W_0 := \begin{array}{c} \bullet \\ s_1 \\ \end{array} \begin{array}{c} \bullet \\ s_2 \\ \end{array} \]

For \( J \subset S \), we denote by \( W_J \) the group generated by \( J \) and by \( w_J \) the longest element of \( W_J \). We look at the subsets \( J \) of \( S \) such that the group generated by \( J \) is isomorphic to \( W_0 \). Here, we find just \( J = \{s_1, s_2\} \) and \( w_J = s_1s_2s_1s_2s_1s_2 \)

Then:

\[ c_0 = \{ w \in W | w = z.w_J.z', \; z, z' \in W \} \]
For $J \subseteq S$, we denote by $W_J$ the group generated by $J$ and by $w_J$ the longest element of $W_J$. We look at the subsets $J$ of $S$ such that the group generated by $J$ is isomorphic to $W_0$. Here, we find just $J = \{s_1, s_2\}$ and $w_J = s_1s_2s_1s_2s_1s_2$. Then:

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Then:

$$c_0 = \{w \in W | w = z.w_J.z', \ z, z' \in W\}$$

Moreover, let $M_J = \{z \in W | sw_Jz \notin c_0, \text{ for all } s \in J\}$. We have:

$$c_0 = \bigcup_{z \in M_J} \{w \in W | w = x.w_J.z, \ x \in W\}$$
We have:

**Theorem. G. (≈ 2007)**

Let \( z \in M_J \). The set \( \{ w \in W \mid w = x.w_J.z, \ x \in W \} \) is a union of left cells.

This implies that:

1. \( c_0 \) contains exactly \( |W_0| \) left cells.
2. For \( z \in M_J \), the set \( \{ w \in W \mid w = x.w_J.z, \ x \in W \} \) is a left cell.
Decomposition in left cells

\[ a \cdot s_1 \cdot b \cdot s_2 \cdot b \cdot s_3, \text{ for all } a > 3b \]