Type-II Solutions to Mean Curvature Flow

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Abstract We survey results on mean curvature flow solutions with Type-II curvature blow up in finite time in hope of understanding the singularity profile.

1. Introduction

Let $\varphi(t) : M^n \to \mathbb{R}^{n+1}$, $t_0 < t < t_1$, be a one-parameter family of immersions of n-dimensional hypersurfaces in the Euclidean space. Mean curvature flow (MCF) evolves the hypersurface M^n in the direction of its mean curvature vector $\vec{H}$ according to the following nonlinear PDE system

$$\partial_t \varphi(p,t) = \vec{H}, \quad p \in M^n, \quad t_t < t < t_1. \tag{1.1}$$

Mean curvature flow is the negative gradient flow of the volume functional. If we denote $M_t := \varphi(t)(M^n)$, then

$$\frac{d}{dt}\mathrm{Vol}(M_t) = -\int_{M_t} H^2.$$

The parablic nature of MCF means that it will smooth out the surface on small time-scale, but over larger time-scale singularity in the solution can occur. For example, consider a round sphere S^n with radius r_0, then under MCF the sphere shrinks according to $r(t) = \sqrt{r_0 - 2nt}$ and collapses to a point in finite time.

The maximum principle for MCF implies that disjoint hypersurfaces remain disjoint under MCF. Because we can always enclose a given compact hypersurface inside a large sphere, MCF of compact hypersurface must have a finite extinction time. Because the round cylinder $S^{n-1} \times \mathbb{R}$ collapses to a line in finite time, MCF starting from a noncompact hypersurface trapped inside a cylinder must have a finite extinction time.

Let $h(p,t)$ denote the second fundamental form at the point $\varphi(p,t) \in M_t$. Suppose we have a MCF solution on a maximal time interval $[0,T)$. In this survey, we always assume $T < \infty$, then $\limsup_{t \nearrow T} \sup_{M_t} |h| = \infty$. We say the solution forms a **Type-I** singularity if $\sup_{M_t \times [0,T)} (T-t)|h|^2 < \infty$ or a **Type-II** singularity if $\sup_{M_t \times [0,T)} (T-t)|h|^2 = \infty$.

When $n = 1$, MCF of (closed) embedded curves always form Type-I singularity [11, 12]. However, if the curve is immersed in the plane, then Angenent and Velázquez constructed solution with Type-II rate $\sqrt{\frac{\ln\ln(1/(T-t))}{T-t}}$, which is faster than $(T-t)^{-1/2}$ but slower than any higher power $(T-t)^{-1/2-\epsilon}$ [2]. In dimension two or higher, Type-II MCF solutions exist for embedded hypersurfaces: compact examples with blow up rates $(T-t)^{-1+1/m}$ for integer $m \geqslant 3$, by Angenent and Velázquez [3]; noncompact examples with blow up rates $(T-t)^{-(1/2+\gamma)}$ for real number $\gamma \geqslant 1/2$ by Isenberg and the authors [18, 19].

To study singularities, we use blow up analysis. For example, suppose MCF has a Type-I singularity, then we can pick a sequence of points and times $\{x_i, t_i\}_{i=0}^{\infty}$ with $t_i \nearrow T$ and $\lambda_i := |h(x_i, t_i)| \nearrow \infty$ as $i \nearrow \infty$. Rescaling the embedding parabolically

$$\varphi_i := \lambda_i \left(\varphi\left(x, \lambda_i^{-2} t + t_i\right) - \varphi\left(x_i, t_i\right)\right),$$

then compactness theorem implies that φ_i converges along a subsequence to a limiting geometry called the (pointed) singularity model. In particular, this limit is non-flat. By Huisken's monotonicity formula [14], Type-I singularities in mean convex (i.e. $H := \mathrm{tr}_g h > 0$, where $g = \varphi^* \delta$ is the pull-back of the Euclidean metric on the hypersurface) MCF are modelled by self-shrinking solution satisfying the elliptic equation

$$H = \langle \varphi, \nu \rangle,$$

where ν is the (outer) unit normal to the hypersurface. If φ_0 is a self-shrinking soliton, then $\sqrt{1-2t}\varphi$ is a homothetically shrinking MCF solution. By a similar blow up, Huisken and Sinestrari [16] showed that Type-II singularities in mean convex MCF are modelled by translating solitons which satisfy

$$H + \langle \nu, \vec{v} \rangle = 0$$

for some constant vector $\vec{v} \in \mathbb{R}^{n+1}$. If φ_0 is a translating soliton, then $\varphi_0 + t\vec{v}$ is a MCF solution.

2. Level Set Flow

Mean convex hypersurfaces M_t evolving by MCF can be represented as the level set of a function v, called the arrival time,

$$M_t = \{x \in \mathbb{R}^{n+1} : v(x) = t\}.$$

Then v satisfies the level set equation

$$\operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right) = -\frac{1}{|\nabla v|}. \tag{1.2}$$

This equation is well studied in the literature, for example, the numerical work by Osher and Sethian [20], and the theoretical justification via the theory of viscosity solutions by Evans and Spruck [10], and independently, by Chen, Giga and Goto [4].

Analytically, the level set equation (1.2) is degenerate elliptic. While solution is known to be Lipschitz [10], a priori it is not even differentiable. In [6], Colding and Minicozzi proved that the the singular set of MCF corresponds to the critical set $\{\nabla v = 0\}$ of v and obtained the following differentiability result for v.

Theorem 1 (Colding-Minicozzi 2016). *For a MCF starting at a closed smooth mean convex hypersurface M_0, the arrival time is twice differentiable everywhere and smooth away from the critical set, and has uniformly bounded second derivative. Moreover:*

- *The critical set is contained in finitely many compact embedded $(n-1)$-dimensional Lipschitz submanifolds plus a set of dimension at most $n-2$.*
- *At each critical point the Hessian is symmetric and has only two eigenvalues 0 and $-\frac{1}{k}$; 0 has multiplicity $n-k$ (which could be 0) and $-\frac{1}{k}$ has multiplicity $k+1$ for some $k \in \{1, 2, \ldots, n\}$.*

- *It satisfies the level set equation* (1.2) *everywhere in the classical sense.*

If the initial hypersurface is convex, then the flow is smooth except at the point it becomes extinct and Huisken showed that the arrival time is C^2 [14, 15]. For curves in the plane, Kohn and Serfaty showed that the arrival time is at least C^3, but for $n \geqslant 2$, Sesum gave examples of convex initial hypersurfaces where the arrival time is not three times differentiable, so Huisken's result is optimal. In [7], Colding and Minicozzi gave a necessary and sufficient condition for u to be C^2.

Theorem 2 (Colding-Minicozzi 2018)**.** *The arrival time is C^2 if and only if both of the following hold:*

(1) There is exactly one singular time T (where MCF becomes extinct).

(2) The singular set $\mathcal{S}$ is a k-dimensional closed connected embedded C^1 submanifold of singularities where the blowup is a cylinder $S^{n-k} \times \mathbb{R}^k$ at each point and $\mathcal{S}$ is tangent to the $\mathbb{R}^k$ factor.

On the other hand, there are rotationally symmetric mean convex dumbbell solutions by Angenent, Altschuler and Giga [1], and Ilmanen [17] for which the arrival time is not C^2.

3. Type-II Examples

While Type-I MCF solutions clearly exist, the existence of Type-II solution is not immediately obvious and often speculated as a critical phenomenon when MCF solutions undergo phase changes. Consider a one-parameter family of S^n ($n \geqslant 2$) with the parameter controlling the amount of cinching at the equator. There are three possible scenarios: (1) for very loose cinching, MCF of convex hypersurface forms a Type-I singularity modelled by the round sphere [13]; (2) for very tight cinching, under MCF the equator shrinks more rapidly than the two "dumbbell" ends, the flow forms a Type-I singularity whose blow up at the equator is a shrinking cylinder soliton; (3) for some critical cinching in between, the flow behaves differently from (1) or (2) and is expected to form a Type-II singularity.

This heuristic picture for Type-II MCF solution was rigorously justified by Angenent, Altschuler and Giga [1] on mean convex rotationally symmetric hypersurfaces. In particular, the solution shrinks to a point without ever becoming convex. Their proof was by contradiction, and did not provide any quantitative information.

Using different method, Angenent and Velázquez [3] constructed Type-II solutions explicitly. Their result can be summarised as follows (for details see [3]).

Theorem 3 (Angenent-Velázquez 1997)**.** *For each integer $m \geqslant 3$, there exists rotationally symmetric MCF solution on S^n $(n \geqslant 2)$ such that*

- *The solution forms a Type-II singularity:* $\sup_{S^n} |h| \sim (T-t)^{-(1-1/m)}$.
- *The singular region for the surface is described by three (overlapping) regions: tip, intermediate and neck. At the neck, the Type-I blow up of the solution converges to a shrinking cylinder soliton. At the tip, it achieves maximum curvature and the Type-II blow up of the solution converges to a translating soliton known as the bowl soliton.*

The case of odd m is the main concern in [3] which corresponds to the asymmetric dumbbell. The case of even m is also considered in [3] as the peanut example. We note that these Type-II blow up rates form a quantised subset of $[2/3, 1)$, which is consistent with the non-genericity of compact Type-II solutions by Colding and Minicozzi [5].

Let us elaborate on the theorem. Suppose the rotationally symmetric is obtained by rotating a curve $r = u(x)$ around the x-axis. Then MCF is:

$$\frac{\partial}{\partial t} u = \frac{u_{xx}}{1+u_x^2} - \frac{n-1}{u}. \tag{1.3}$$

The result says the highest curvature occurs at the tip (where $u = 0$) and is of order $(T-t)^{-(1-1/m)}$. The axial direction (x-axis) for the tip region is roughly of size $(T-t)^{1-\frac{1}{m}}$. The geometry looks like a translating bowl soliton scaled down by $(T-t)^{-1+\frac{1}{m}}$. In the neck region (around the equator of the sphere), the curvature is $(T-t)^{-\frac{1}{2}}$, the radius shrinks at the rate of $(T-t)^{\frac{1}{2}}$, and the axial direction is roughly of size $(T-t)^{\frac{1}{2}}$. The geometry looks like a shrinking cylinder with decaying error given by the m-th Hermite polynomial.

The intermediate region is the transition region between the tip and the neck. The axial direction is roughly of size $(T-t)^{\frac{1}{m}}$. The rescaled geometry looks like the rotation of the profile curve given by $\phi^2 + Ky^m = 2(n-1)$ with constant $K > 0$. To formally explain this profile, let us introduce the following multi-scaled parameters

$$\tau = -\log(T-t), \quad y = x(T-t)^{-1/m}, \quad \phi(y,\tau) = u(x,t)(T-t)^{-1/2},$$

then Equation (1.3) becomes

$$\phi_\tau = \frac{1}{2}\phi - \frac{y}{m}\phi_y - \frac{n-1}{\phi} + \frac{e^{(-1+\frac{2}{m})\tau}\phi_{yy}}{1+e^{(-1+\frac{2}{m})\tau}\phi_y^2}. \tag{1.4}$$

Assuming there is a strong limit $\phi(y)$ as $\tau \nearrow \infty$, it must satisfy the limiting equation

$$\frac{1}{2}\phi - \frac{y}{m}\phi_y - \frac{n-1}{\phi} = 0, \tag{1.5}$$

whose solution is nothing but the profile curve. For the case of even m with the surface being mirror symmetric, we see that the MCF solution converges to a point, or an interval if rescaled to have fixed diameter. We call this the **singularity profile** in contrast with the notion of singularity model mentioned before, which is at the maximum curvature locus. The above description says that the tip region (of maximum curvature) is much smaller than the rest, and so one should not merely restrict to it when trying to describe the global geometry.

The existence of Type-II MCF solutions can also be posed for non-compact hypersurfaces. Consider a smooth complete convex graph over the ball B^n such that the graph is asymptotic to the cylinder $\partial B^n \times \mathbb{R} = S^{n-1} \times \mathbb{R}$. Under MCF, the graph remains smooth and asymptotic to the contracting cylinder, and disappears at spatial infinity in finite time T, the same time when the cylinder collapses. In the work [21] by Sáez-Schnürer, there is general discussion of this picture in the analytic setting. Because the hypersurface must travel infinite distance in finite amount of time and the speed is proportional to the curvature, we expect the solution to have "fast" (Type-II) curvature blow up. Again, let us work with rotationally symmetric hypersurfaces for which the MCF equation is given in (1.3). Introducing the following rescaled quantities

$$\tau = -\log(T-t), \quad y = x(T-t)^{\gamma-1/2}, \quad \phi(y,\tau) := u(x,t)(T-t)^{-1/2},$$

the first author and Isenberg proved the following result in [18].

Theorem 4 (Isenberg-Wu 2019)**.** *For each real number $\gamma > 1/2$, there exists a family of MCFs of complete, strictly convex, graphical hypersurfaces in $\mathbb{R}^{n+1}$ ($n \geqslant$*

2) over a (shrinking) n-ball and inside a cylinder s.t. each such hypersurface Γ_t escapes at spatial infinity while the cylinder becomes singular in finite time $T < \infty$. The precise asymptotic properties towards time T are the following:

- *The highest curvature occurs at the tip with Type-II rate*

$$\sup_{\Gamma_t} |h| \sim (T-t)^{-(\gamma+1/2)} \quad \text{as } t \nearrow T.$$

- *Near the tip, the Type-II blow up of Γ_t converges to a bowl soliton.*
- *Near spatial infinity, the Type-I blow up of Γ_t is asymptotic to a cylinder at a precise rate depending on γ:*

$$2(n-1) - \phi^2 \sim y^{\frac{1}{1/2-\gamma}} \quad \text{as } y \nearrow \infty.$$

We note that these Type-II blow up rates form a continuum $(1, \infty)$. Comparing this result with that by Angenent-Velázquez, we see that the parameter $-\gamma - 1/2$ corresponds to $-1+1/m$ as the rate of curvature blow up, and $\gamma - 1/2$ corresponds to $-1/m$ as the scaling from x to y. While there are essential differences in the non-compact solutions with $\gamma > 1/2$ and the compact solutions with $m \geqslant 3$, if we let $\gamma \to 1/2^+$ and $m \to \infty$, then we arrive at the Type-II curvature blow up rate $(T-t)^{-1}$, which might be considered as a "critical" Type-II solution. The analysis in [18] does not carry through to the limit as $\gamma \to 1/2$. Instead, we work with the following rescalings

$$\tau = -\log(T-t), \quad y = x + a\log(T-t), \quad \phi(y,\tau) := u(x,t)(T-t)^{-1/2},$$

Then, in a joint work with Isenberg, we have the following result corresponding to the critical case $\gamma = 1/2$ in [19].

Theorem 5 (Isenberg-Wu-Zhang 2019)**.** *There exists a family of MCFs of complete, strictly convex, graphical hypersurfaces in $\mathbb{R}^{n+1}$ ($n \geqslant 2$) over a (shrinking) n-ball and inside a cylinder s.t. each such hypersurface Γ_t escapes at spatial infinity while the cylinder becomes singular in finite time $T < \infty$. The precise asymptotic properties near time T are the following:*

- *The highest curvature occurs at the tip with Type-II rate*

$$\sup_{\Gamma_t} |h| \sim (T-t)^{-1} \quad ast \nearrow T.$$

- *Near the tip, the Type-II blow up of* Γ_t *converges to a bowl soliton.*
- *Near spatial infinity, the Type-I blow up of* Γ_t *is asymptotic to a cylinder at a precise rate depending on* γ*:*

$$2(n-1) - \phi^2 \sim e^{-y} \quad \textit{as } y \nearrow \infty.$$

4. Local Analysis for Symmetric Type-II Solutions

In this section, we carry out local analysis for the mirror symmetric Type-II solutions with examples in [3] (for even $m > 2$). For them, MCF shrinks the hypersurface to a non-convex point, where in light of the results in [6] and [7] on the level set flow, all the singular information will be coded. Thus we consider the Taylor expansion at $y = 0$ for the scaled picture described by Equation (1.4). This can be understood as a step to justify the convergence to the singularity profile.

Before diving into the analysis, we provide some heuristic arguments as motivation. If there is a strong smooth limit $\phi(y)$ as $\tau \to \infty$. Then the limit $\phi(y)$ must satisfy the following limiting equation (1.5):

$$\frac{1}{2}\phi - \frac{y}{m}\phi_y - \frac{n-1}{\phi} = 0,$$

and we solve it to get the profile curve for the singularity as follows. Start with

$$\phi^2 - \frac{2}{m} y \phi \phi_y = 2(n-1)$$

and set $V = \phi^2 - 2(n-1)$. Then

$$V - \frac{y}{m} V_y = 0, \quad \text{then} \quad \frac{V_y}{V} = \frac{m}{y}, \quad \text{so} \quad \log|V| = m \log|y| + C.$$

Thus $V + K y^m = 0$, and so

$$\phi^2 + K y^m = 2(n-1) \tag{1.6}$$

for some constant K. Since ϕ needs to be 0 at the tips, K is positive. This is exactly the function used to construct the intermediate region in [3].

Now we investigate the dynamic property of the scaled evolution equation (1.4), i.e. the possible convergence to the singularity profile (1.6).

First, we make the following observation. For Equation (1.4), if we could control ϕ and its derivatives uniformly near $y = 0$, then formally the equation could be simplified to

$$v_\tau = \frac{1}{2}v - \frac{y}{m}v_y - \frac{n-1}{v},$$

but the limiting behaviour of v is unstable at $y = 0$, as explained below. One restricts to the case of v being even as is clearly preserved the simplified flow. Considering the value at $y = 0$, $A = v(0, \tau)$ satisfies

$$A_\tau = \frac{1}{2}A - \frac{n-1}{A}.$$

We see $A = \sqrt{2(n-1)}$ is an unstable equilibrium solution for this ODE, which indicates the delicate nature of the problem.

Now we study the Taylor expansion of $\phi(y, \tau)$ at $y = 0$. We impose the **Boundedness Assumption** (simplified as BA): $\phi_{yy}(y, \tau)$ is uniformly bounded for $\tau \in [0, \infty)$ in a small neighbourhood of y around $y = 0$, or all y-derivatives of ϕ at $y = 0$ are uniformly bounded for $\tau \in [0, \infty)$. Obviously, it implies that the curvature is of Type-I rate at $y = 0$. We point out that the examples in [3] all satisfy this assumption.

The analyticity with respect to y is standard. Since $\phi_y(0, \tau) = 0$ by evenness, for $r(\tau) = \phi(0, \tau)$, (1.4) gives

$$r_\tau = \frac{1}{2}r - \frac{n-1}{r} + e^{(-1+\frac{2}{m})\tau}\phi_{yy}(0, \tau).$$

By BA, r and $\phi_{yy}(0, \tau)$ are uniformly bounded. So we have

$$[r^2 - 2(n-1)]_\tau = [r^2 - 2(n-1)] + O\left(e^{(-1+\frac{2}{m})\tau}\right),$$

$$[e^{-\tau}\left(r^2 - 2(n-1)\right)]_\tau = O\left(e^{(-2+\frac{2}{m})\tau}\right),$$

$$e^{-\tau}\left(r(\tau)^2 - 2(n-1)\right) - e^{-S}\left(r(S)^2 - 2(n-1)\right) = O(1)\left(e^{(-2+\frac{2}{m})\tau} - e^{(-2+\frac{2}{m})S}\right).$$

Noticing $m > 2$, we let $S \to \infty$ and apply the boundedness of r to arrive at

$$r^2 - 2(n-1) = O\left(e^{(-1+\frac{2}{m})\tau}\right),$$

and conclude

$$r = \sqrt{2(n-1)} + O\left(e^{(-1+\frac{2}{m})\tau}\right).$$

It's also clear from the equation that

$$r_\tau = O\left(e^{(-1+\frac{2}{m})\tau}\right).$$

Now we consider (1.4) using the Taylor expansion at $y = 0$. Set

$$\phi(y,\tau) = r(\tau) + \sum_{k=1}^{\infty} f_k(\tau) y^{2k}, \tag{1.7}$$

and we have

$$\phi_y = \sum_{k=1}^{\infty} 2k f_k(\tau) y^{2k-1},$$

$$\phi_{yy} = \sum_{k=1}^{\infty} 2k(2k-1) f_k(\tau) y^{2k-2},$$

$$\phi^2 - 2(n-1) = r^2 - 2(n-1) + 2r \sum_{k=1}^{\infty} f_k(\tau) y^{2k} + \left(\sum_{k=1}^{\infty} f_k(\tau) y^{2k}\right)^2.$$

So (1.4) becomes

$$[\phi^2 - 2(n-1)]_\tau = [\phi^2 - 2(n-1)] - \frac{y}{m}[\phi^2 - 2(n-1)]_y + \frac{2e^{(-1+\frac{2}{m})\tau}\phi\phi_{yy}}{1 + e^{(-1+\frac{2}{m})\tau}\phi_y^2}$$

which is then, denoting $\frac{d}{d\tau}$ by $'$,

$$\begin{aligned}
&2rr' + 2r' \sum_{k=1}^{\infty} f_k(\tau) y^{2k} + 2r \sum_{k=1}^{\infty} f_k'(\tau) y^{2k} + 2 \sum_{k=1}^{\infty} f_k(\tau) y^{2k} \cdot \sum_{k=1}^{\infty} f_k'(\tau) y^{2k} \\
= &\, r^2 - 2(n-1) + 2r \sum_{k=1}^{\infty} f_k(\tau) y^{2k} + \left(\sum_{k=1}^{\infty} f_k(\tau) y^{2k}\right)^2 \\
&- 2r \sum_{k=1}^{\infty} \frac{2k}{m} f_k(\tau) y^{2k} - 2 \sum_{k=1}^{\infty} f_k(\tau) y^{2k} \cdot \sum_{k=1}^{\infty} \frac{2k}{m} f_k(\tau) y^{2k} \\
&+ \frac{2e^{(-1+\frac{2}{m})\tau}\phi\phi_{yy}}{1 + e^{(-1+\frac{2}{m})\tau}\phi_y^2}.
\end{aligned} \tag{1.8}$$

We now analyse f_k's by comparing the coefficients for powers of y. Recall that $m > 2$ is an even interger.

1. For the constant term: we have

$$2rr' = r^2 - 2(n-1) + 2e^{(-1+\frac{2}{m})\tau} r \cdot 2f_1 \cdot 1.$$

   By the previous obtained expressions of r and r' and the boundedness of f_1, i.e. $\phi_{yy}(0)$, both sides are terms like $O\left(e^{(-1+\frac{2}{m})\tau}\right)$.

2. For y^2 term: we point out that, up to a term of $O\left(e^{(-1+\frac{2}{m})\tau}\right)$, in light of BA, we can ignore that last term on the right have side of (1.4). Since one can take y-derivatives for the above equation on series and set $y=0$ for equivalent discussion, we can use either version of BA for this simplification. Thus we have

$$2r'f_1 + 2rf_1' = 2rf_1 - \frac{4r}{m}f_1 + O\left(e^{(-1+\frac{2}{m})\tau}\right).$$

By BA and the expression of r', we have

$$f_1' = \left(1-\frac{2}{m}\right)f_1 + O\left(e^{(-1+\frac{2}{m})\tau}\right),$$

$$[e^{(-1+\frac{2}{m})\tau}f_1]_\tau = O\left(e^{(-2+\frac{4}{m})\tau}\right).$$

Integrate over $[\tau, S]$ to arrive at

$$e^{(-1+\frac{2}{m})\tau}f_1(\tau) - e^{(-1+\frac{2}{m})S}f_1(S) = O(1)\left(e^{(-2+\frac{4}{m})\tau} - e^{(-2+\frac{4}{m})S}\right)$$

Letting $S\to\infty$ and applying the boundedness of f_1, we conclude

$$f_1 = O\left(e^{(-1+\frac{2}{m})\tau}\right)$$

and by the above ODE evolution,

$$f_1' = O\left(e^{(-1+\frac{2}{m})\tau}\right).$$

Note: since $1-\frac{2}{m}>0$, if we ignore the remainder in the ODE evolution of f_1, $f_1(0)$ clearly has to vanish for the boundedness. This is the major difference in the study of f_k for $k<\frac{m}{2}$ and otherwise.

3. For y^4 term: similarly absorbing controlled terms to the remainder, we have

$$2rf_2' = 2rf_2 - \frac{8r}{m}f_2 + O\left(e^{(-1+\frac{2}{m})\tau}\right),$$

$$f_2' = \left(1-\frac{4}{m}\right)f_2 + O\left(e^{(-1+\frac{2}{m})\tau}\right),$$

$$[e^{(-1+\frac{4}{m})\tau}f_2]_\tau = O\left(e^{(-2+\frac{6}{m})\tau}\right).$$

Then we consider the following cases:

- $m > 4$: we proceed as for y^2 term and integrate over $[\tau, S]$ to get

$$e^{(-1+\frac{4}{m})\tau} f_2(\tau) - e^{(-1+\frac{4}{m})S} f_2(S) = O(1)\left(e^{(-2+\frac{6}{m})\tau} - e^{(-2+\frac{6}{m})S}\right).$$

  Letting $S \to \infty$ and by the boundedness of f_2 from BA, we conclude

$$f_2 = O\left(e^{(-1+\frac{2}{m})\tau}\right), \quad f_2' = O\left(e^{(-1+\frac{2}{m})\tau}\right).$$

- $m = 4$: we have

$$f_2' = O\left(e^{(-1+\frac{2}{m})\tau}\right).$$

  Then by integration, one has $f_2(S) \to A_1$ for some A_1 as $S \to \infty$ and

$$f_2 = A_1 + O\left(e^{(-1+\frac{2}{m})\tau}\right).$$

4. Suppose we have done for y^{2K}, and now consider $y^{2(K+1)}$ term.

- $2K < m$: by induction, we have $f_k' = O(e^{-\epsilon\tau})$ and $f_k = O(e^{-\epsilon\tau})$ for some $\epsilon > 0$ and $k = 1, \cdots, K$, which take care of the product terms in (1.8) and we have:

$$2rf_{K+1}' = 2rf_{K+1} - \frac{4r(K+1)}{m} f_{K+1} + O(e^{-\epsilon\tau}), \tag{1.9}$$

$$f_{K+1}' = \left(1 - \frac{2(K+1)}{m}\right) f_{K+1} + O(e^{-\epsilon\tau}).$$

  Now we can recycle the consideration for y^4 term. For some (possibly different) $\epsilon > 0$:

  (a) If $2(K+1) < m$, recycling the treatment for the case of $m > 4$ for y^4, we conclude

$$f_{K+1}' = O(e^{-\epsilon\tau}), \quad f_{K+1} = O(e^{-\epsilon\tau}).$$

  (b) If $2(K+1) = m$, then as in the case of $m = 4$ for y^4, we conclude

$$f_{K+1}' = O(e^{-\epsilon\tau}), \quad f_{K+1} = A_1 + O(e^{-\epsilon\tau}).$$

- $2K = m$: by induction, for some $\epsilon > 0$, we have $f_k' = O(e^{-\epsilon\tau})$ for $k = 1, \cdots, K$ and $f_k = O(e^{-\epsilon\tau})$ for $k = 1, \cdots, K-1$ and $f_K = A_1 + O(e^{-\epsilon\tau})$. Then for $K+1$, the left hand side from (1.8) remains the same since the relevant f_k' is always $O(e^{-\epsilon\tau})$. There could be extra terms on the right hand side coming from the product terms, but it only happens when $2(K+1)$ is a multiple of m, which is not possible since $2K = m$. So the equation is the same as (1.9):

$$2rf'_{K+1} = 2rf_{K+1} - \frac{4r(K+1)}{m} f_{K+1} + O(e^{-\epsilon\tau}),$$

$$f'_{K+1} = \left(1 - \frac{2(K+1)}{m}\right) f_{K+1} + O(e^{-\epsilon\tau})$$

but now we have $m = 2K < 2(K+1)$. Then we just need to integrate over $[0, \tau]$ to conclude

$$f'_{K+1} = O(e^{-\epsilon\tau}), \quad f_{K+1} = O(e^{-\epsilon\tau})$$

without making use of BA.

- $2K > m$: the equation for f_{K+1} remains the same as (1.9) until we reach $2(K+1) = 2m$. Then for y^{2m} term, there is an extra constant A_2, in comparison to (1.9), coming from the product terms:

$$f'_{K+1} = -f_{K+1} + A_2 + O(e^{-\epsilon\tau}). \tag{1.10}$$

By considering $f_{K+1} - A_2$ instead, we see

$$f_{K+1} = A_2 + O(e^{-\epsilon\tau}), \quad f'_{K+1} = O(e^{-\epsilon\tau}).$$

without any assumption.

This is for f_k when k is an integer multiple of $\frac{m}{2}$.

Finally, we summarise the above in the following theorem.

Theorem 6. *Let the constant m be an even integer greater than 2. Impose* **Boundedness Assumption** *and consider the Taylor expansion of the solution ϕ (1.7) at $y = 0$ to (1.4) for all time. For some constant $\epsilon > 0$ which might change from line to line,*

$$f_k = O(e^{-\epsilon\tau}), \quad f'_k = O(e^{-\epsilon\tau});$$

except that for constants A_μ for $\mu = 1, 2, \cdots$,

$$f_{\frac{\mu m}{2}} = A_\mu + O(e^{-\epsilon\tau}).$$

Remark 7. *When $k < \frac{m}{2}$, after ignoring the remainder term, the evolution of f_k says that the initial value has to vanish for it to stay bounded for all τ, as a hint for the bad impact of lower power terms in the construction. Intuitively, the term y^{2k} for small k provides a "steeper" neck in the middle, i.e. with value changing at a faster rate when leaving* 0. *So by the result in [1], a neck pinch will likely appear with a perturbation of any small size.*

The pattern of the coefficients described in the above theorem is also compatible with the level set flow picture and fits the formal limit $\phi^2 + Ky^m = 2(n-1)$ very well. Namely, for the level set flow, the scaling is $(x,u) = \big(y(T-t)^{1/m}, \phi(T-t)^{1/2}\big)$. Assume the level set function is $V(x,u)$ and we have

$$V(x,u) = V\big(y(T-t)^{1/m}, \phi(T-t)^{1/2}\big) = T-t.$$

The powers y^m and ϕ^2 naturally come up for the leading term consideration.

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