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where $F(x)$ is an anti-derivative of $f(x)$. More about this later…
Consider the functions

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The RHS is the most general anti-derivative of \( f(x) \).

The process of finding \( F(x) \), given \( f(x) \), is called anti-differentiation, or (indefinite) integration.
Differentiation rules — such as $\frac{d}{dx}(x^n) = nx^{n-1}$ — give rise to rules for integration.
General formulae for anti-derivatives

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### Formulae

\[
\int k \, dx = kx + C \quad \text{\( k \) constant}
\]

\[
\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad \text{\( n \neq -1 \)}
\]

\[
\int \frac{1}{x} \, dx = \ln x + C
\]

\[
\int \cos x \, dx = \sin x + C
\]

\[
\int \sin x \, dx = -\cos x + C
\]

\[
\int e^x \, dx = e^x + C
\]
General formulae for anti-derivatives

Formulae (continued)

\[
\int \sec^2 x \, dx = \tan x + C \\
\int \frac{1}{x^2 + 1} \, dx = \tan^{-1} x + C \\
\int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x + C \\
\int \frac{1}{\sqrt{1 + x^2}} \, dx = \ln \left( x + \sqrt{x^2 + 1} \right) + C
\]
General formulae for anti-derivatives

To find anti-derivatives of more complicated functions, we may use the following rules.
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**Note:** No simple rules exist for \( \int f(x)g(x) \, dx \) or \( \int \frac{f(x)}{g(x)} \, dx \).
Exercises

Evaluate the following indefinite integrals.

1. $\int (x^2 + \cos x) \, dx$

2. $\int \left( \frac{2}{x^2 + 1} - 3e^x \right) \, dx$

3. $\int (ax^2 + bx + c) \, dx$, where $a, b, c$ are constants
Suppose $F(x)$ is an anti-derivative for $f(x)$. 
Integration by substitution

Suppose $F(x)$ is an anti-derivative for $f(x)$. Now consider the function

$$y = F(u) \quad \text{where} \quad u = g(x).$$
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$$\frac{d}{dx} (F(u)) = F'(u) \frac{du}{dx} = f(u) \frac{du}{dx}.$$

So

$$\int f(u) \frac{du}{dx} \, dx = F(u) + C.$$
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Integration by substitution

\[ \int f(u) \frac{du}{dx} \, dx = \int f(u) \, du, \]
where \( u \) is a function of \( x \).

Put differently:
Rewriting this last equation, we have the following.

\[ \int f(u) \frac{du}{dx} \, dx = \int f(u) \, du, \quad \text{where } u \text{ is a function of } x. \]

Put differently:

\[ \int f(g(x)) \, g'(x) \, dx = F(g(x)) + C, \]

where \( F(t) \) is an anti-derivative of \( f(t) \).
Exercise

Evaluate the following indefinite integral:

\[ \int e^{x^3} 3x^2 \, dx \]
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Solution

*Let* \( u = x^3 \ldots \)
Exercises

Evaluate the following indefinite integrals:

1. \[ \int \sin^4 x \cos x \, dx \]

2. \[ \int \cos^4 x \sin x \, dx \]

3. \[ \int \tan x \, dx \]

4. \[ \int \frac{\ln x}{x} \, dx \]

5. \[ \int \sqrt{1 + \sqrt{x}} \, dx \]