Roughly speaking, integration is the opposite of differentiation.

- differential calculus concerns slopes of graphs,
- integral calculus concerns areas under graphs.

They are linked by the Fundamental Theorem of Calculus:

\[ \int_a^b f(x) \, dx = F(b) - F(a) \]

where \( F(x) \) is an anti-derivative of \( f(x) \). More about this later...
Consider the functions

\[ F(x) = x^3 \quad \text{and} \quad f(x) = 3x^2. \]

We know that \( F'(x) = f(x) \). We say:

- \( f(x) \) is the derivative of \( F(x) \),
- \( F(x) \) is an anti-derivative of \( f(x) \).

Note that:

- \( \frac{d}{dx} (x^3 + 5) = 3x^2 \) also,
- so \( x^3 + 5 \) is another anti-derivative of \( f(x) \).
Anti-derivatives

In general, if $f(x)$ is any function and $F(x)$ any anti-derivative of $f(x)$, then every anti-derivative of $f(x)$ is of the form $F(x) + C$, where $C$ is some constant.

We write this as:

$$\int f(x) \, dx = F(x) + C.$$

- $\int f(x) \, dx$ is read as “the indefinite integral of $f(x)$ with respect to $x$”.
- The RHS is the most general anti-derivative of $f(x)$.
- The process of finding $F(x)$, given $f(x)$, is called anti-differentiation, or (indefinite) integration.
General formulae for anti-derivatives

Differentiation rules — such as \( \frac{d}{dx} (x^n) = nx^{n-1} \) — give rise to rules for integration.

**Formulae**

\[
\begin{align*}
\int k \, dx &= kx + C & k \text{ constant} \\
\int x^n \, dx &= \frac{x^{n+1}}{n+1} + C & n \neq -1 \\
\int \frac{1}{x} \, dx &= \ln x + C \\
\int \cos x \, dx &= \sin x + C \\
\int \sin x \, dx &= -\cos x + C \\
\int e^x \, dx &= e^x + C
\end{align*}
\]
Formulae (continued)

\[
\int \sec^2 x \, dx = \tan x + C \\
\int \frac{1}{x^2 + 1} \, dx = \tan^{-1} x + C \\
\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C \\
\int \frac{1}{\sqrt{1+x^2}} \, dx = \ln \left( x + \sqrt{x^2 + 1} \right) + C
\]
General formulae for anti-derivatives

To find anti-derivatives of more complicated functions, we may use the following rules.

### Anti-differentiation rules

1. \[
\int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx,
\]
   — the sum/difference law,

2. \[
\int k f(x) \, dx = k \int f(x) \, dx,
\]
   — the scalar multiple law.

**Note:** No simple rules exist for \[
\int f(x)g(x) \, dx \quad \text{or} \quad \int \frac{f(x)}{g(x)} \, dx.
\]
Exercises

Evaluate the following indefinite integrals.

1. \( \int (x^2 + \cos x) \, dx \)

2. \( \int \left( \frac{2}{x^2 + 1} - 3e^x \right) \, dx \)

3. \( \int (ax^2 + bx + c) \, dx \), where \( a, b, c \) are constants
Suppose $F(x)$ is an anti-derivative for $f(x)$. Now consider the function
\[ y = F(u) \quad \text{ where } u = g(x). \]

The chain rule for differentiation says:
\[ \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}. \]

Rewriting this, we have
\[ \frac{d}{dx} (F(u)) = F'(u) \frac{du}{dx} = f(u) \frac{du}{dx}. \]

So
\[ \int f(u) \frac{du}{dx} \, dx = F(u) + C. \]
Rewriting this last equation, we have the following.

Integration by substitution

$$\int f(u) \frac{du}{dx} \, dx = \int f(u) \, du, \quad \text{where } u \text{ is a function of } x.$$  

Put differently:

Integration by substitution

$$\int f(g(x)) g'(x) \, dx = F(g(x)) + C,$$

where $F(t)$ is an anti-derivative of $f(t)$.  

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Exercise

Evaluate the following indefinite integral:

\[ \int e^{x^3} 3x^2 \, dx \]

Solution

Let \( u = x^3 \) \ldots
Exercises

Evaluate the following indefinite integrals:

1. \( \int \sin^4 x \cos x \, dx \)
2. \( \int \cos^4 x \sin x \, dx \)
3. \( \int \tan x \, dx \)
4. \( \int \frac{\ln x}{x} \, dx \)
5. \( \int \sqrt{1 + \sqrt{x}} \, dx \)