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**Examples**

- $f(x) = x^2$, or $f(x) = x^n$ where $n$ is any **even** integer
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**Examples**

- $f(x) = x^2$, or $f(x) = x^n$ where $n$ is any **even** integer
- $f(x) = \cos x$
Even and odd functions

A function \( f(x) \) is an odd function if

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f(-x) = -f(x) \quad \text{for all } x.
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Even and odd functions

A function $f(x)$ is an **odd function** if

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**Examples**

- $f(x) = x$, 

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**Examples**

- $f(x) = x$, or $f(x) = x^n$ where $n$ is any *odd* integer
A function $f(x)$ is an **odd function** if

$$f(-x) = -f(x) \quad \text{for all } x.$$ 

**Examples**

- $f(x) = x$, or $f(x) = x^n$ where $n$ is any **odd** integer
- $f(x) = \sin x$
Even and odd functions

Rules
If \( f(x) \) and \( g(x) \) are both even, then \( f(x) \pm g(x) \) are both even.
Even and odd functions

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1. If $f(x)$ and $g(x)$ are both even, then $f(x) \pm g(x)$ are both even.

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Rules

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Even and odd functions

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4. If one of $f(x)$ and $g(x)$ is even, and the other odd, then $f(x)g(x)$ and $\frac{f(x)}{g(x)}$ are both odd.
Even and odd functions

Rule

If \( f(x) \) is an even function, then
\[
\int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx.
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Even and odd functions

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If \( f(x) \) is an even function, then 
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Rule

If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x) \, dx = 0$. 
Even and odd functions

Rule

If \( f(x) \) is an odd function, then \[ \int_{-a}^{a} f(x) \, dx = 0. \]
Even and odd functions

Exercises

Calculate the following integrals by exploiting properties of even/odd functions.

1.1 \( \int_{-1}^{1} x^{99} \, dx \)

1.2 \( \int_{-\pi/2}^{\pi/2} \cos x \, dx \)

1.3 \( \int_{-2}^{2} (x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) \, dx \)

1.4 \( \int_{-\pi/4}^{\pi/4} \left( \tan^2 x \sec^2 x + \frac{x \cos x}{1 + x^6} \right) \, dx \)
If $f(t)$ represents a rate of change
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$$\int_{a}^{b} f(t) \, dt$$

is equal to the total change between times $t = a$ and $t = b$. 
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is equal to the total change between times \( t = a \) and \( t = b \). Why?

**Exercise**

2 Suppose the velocity of a car (measured in \( \text{ms}^{-1} \)) is given by

\[
v(t) = 28(1 - e^{-0.1t}),
\]

where \( t \) represents time, measured in seconds. Calculate the total distance travelled in the first minute.
Calculating averages
The average value of a function $f(t)$ over an interval $a \leq t \leq b$ is given by
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Why?
Calculating averages

The average value of a function \( f(t) \) over an interval \( a \leq t \leq b \) is given by

\[
\frac{1}{b-a} \int_a^b f(t) \, dt.
\]

Why?

Exercises

3.1 Calculate the average velocity of the car in the previous exercise, over the first minute.

3.2 Calculate the average value of the function \( y = x^2 \) over the interval \( 0 \leq x \leq 4 \).
Suppose $f(x) \geq 0$ for $a \leq x \leq b$. 
Volumes of revolution

Suppose $f(x) \geq 0$ for $a \leq x \leq b$.

- Consider the region bounded by $y = f(x)$, the $x$-axis, and the lines $y = a$ and $y = b$. 
Volumes of revolution

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Volumes of revolution

Suppose $f(x) \geq 0$ for $a \leq x \leq b$.

- Consider the region bounded by $y = f(x)$, the $x$-axis, and the lines $y = a$ and $y = b$.
- If this region is rotated around the $x$-axis, we obtain a solid.
Volumes of revolution

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- The volume of this solid is given by \( \pi \int_a^b f(x)^2 \, dx \).
Volumes of revolution

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- Why?
Volumes of revolution

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- Consider the region bounded by $y = f(x)$, the $x$-axis, and the lines $y = a$ and $y = b$.
- If this region is rotated around the $x$-axis, we obtain a solid.
- The volume of this solid is given by $\pi \int_a^b f(x)^2 \, dx$.

Why? The answer relies on Riemann sums.
Riemann sums
Riemann sums

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Partition the interval up into \( n \) sub-intervals of width \( \Delta x \):
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Partition the interval up into $n$ sub-intervals of width $\Delta x$: 

$$y = f(x)$$
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Partition the interval up into \( n \) sub-intervals of width \( \Delta x \):

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a = x_1 < x_2 < x_3 < \cdots < x_n < x_{n+1} = b.
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The area of the rectangles shown is equal to $\sum_{i=1}^{n} f(x_i)\Delta x$. 

$$y = f(x)$$

$a = x_1 \quad x_2 \quad x_3 \quad x_4 \quad \cdots \quad x_n \quad b = x_{n+1}$
Riemann sums

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- Partition the interval up into \( n \) sub-intervals of width \( \Delta x \):
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a = x_1 < x_2 < x_3 < \cdots < x_n < x_{n+1} = b.
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This is a Riemann sum.
Consider a function $f(x)$ defined on an interval $a \leq x \leq b$. Partition the interval up into $n$ sub-intervals of width $\Delta x$:

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The area of the rectangles shown is equal to $\sum_{i=1}^{n} f(x_i)\Delta x$.

This is a Riemann sum.

As $n \to \infty$, this approximation approaches $\int_{a}^{b} f(x) \, dx$. 

$$y = f(x)$$
By using Riemann sums, we are able to calculate volumes of other solids of revolution.
Other volumes

By using Riemann sums, we are able to calculate volumes of other solids of revolution.

Exercise

4.1 Sketch the region $R$ bounded by the curve $y = -(x - 1)(x - 3)$ and the $x$-axis.

4.2 A solid $S$ is obtained by rotating $R$ about the $y$-axis. Sketch $S$.

4.3 Consider a vertical strip under the curve at some point $1 \leq x_i \leq 3$ of width $\Delta x$, which is rotated about the $y$-axis to form a shell. Sketch the shell.

4.4 Imagine the shell being cut vertically and opened out flat. Find the volume of the shell.

4.5 Using a Riemann sum and a limit, express the volume of $S$ as an integral. Hence find the volume of $S$. 