A function $f(x)$ is an **even function** if

$$f(-x) = f(x) \quad \text{for all } x.$$
A function $f(x)$ is an odd function if

$$f(-x) = -f(x) \quad \text{for all } x.$$
Even and odd functions

Rules

1. If $f(x)$ and $g(x)$ are both even, then $f(x) \pm g(x)$ are both even.

2. If $f(x)$ and $g(x)$ are both odd, then $f(x) \pm g(x)$ are both odd.

3. If $f(x)$ and $g(x)$ are both even, or both odd, then $f(x)g(x)$ and \( \frac{f(x)}{g(x)} \) are both even.

4. If one of $f(x)$ and $g(x)$ is even, and the other odd, then $f(x)g(x)$ and \( \frac{f(x)}{g(x)} \) are both odd.
Even and odd functions

Rule

If $f(x)$ is an even function, then $\int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx$. 
If \( f(x) \) is an odd function, then \[
\int_{-a}^{a} f(x) \, dx = 0.
\]
Even and odd functions

Exercises

Calculate the following integrals by exploiting properties of even/odd functions.

1.1 \[ \int_{-1}^{1} x^{99} \, dx \]

1.2 \[ \int_{-\pi/2}^{\pi/2} \cos x \, dx \]

1.3 \[ \int_{-2}^{2} (x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) \, dx \]

1.4 \[ \int_{-\pi/4}^{\pi/4} \left( \tan^2 x \sec^2 x + \frac{x \cos x}{1 + x^6} \right) \, dx \]
If $f(t)$ represents a rate of change (eg. a velocity, acceleration, growth rate, absorption rate, etc.) then

$$\int_a^b f(t) \, dt$$

is equal to the total change between times $t = a$ and $t = b$. Why?

**Exercise**

2. Suppose the velocity of a car (measured in ms$^{-1}$) is given by

$$v(t) = 28(1 - e^{-0.1t}),$$

where $t$ represents time, measured in seconds. Calculate the total distance travelled in the first minute.
Calculating averages

The average value of a function $f(t)$ over an interval $a \leq t \leq b$ is given by

$$\frac{1}{b-a} \int_a^b f(t) \, dt.$$

Why?

Exercises

3.1 Calculate the average velocity of the car in the previous exercise, over the first minute.

3.2 Calculate the average value of the function $y = x^2$ over the interval $0 \leq x \leq 4$. 
Volumes of revolution

Suppose \( f(x) \geq 0 \) for \( a \leq x \leq b \).

- Consider the region bounded by \( y = f(x) \), the \( x \)-axis, and the lines \( y = a \) and \( y = b \).
- If this region is rotated around the \( x \)-axis, we obtain a solid.
- The volume of this solid is given by \( \pi \int_{a}^{b} f(x)^2 \, dx \).

**Why?** The answer relies on **Riemann sums**.
• Consider a function \( f(x) \) defined on an interval \( a \leq x \leq b \).

• Partition the interval up into \( n \) sub-intervals of width \( \Delta x \):

\[
a = x_1 < x_2 < x_3 < \cdots < x_n < x_{n+1} = b.
\]

• The area of the rectangles shown is equal to \( \sum_{i=1}^{n} f(x_i)\Delta x \).

This is a Riemann sum.

• As \( n \to \infty \), this approximation approaches \( \int_{a}^{b} f(x) \, dx \).
By using Riemann sums, we are able to calculate volumes of other solids of revolution.

Exercise

4.1 Sketch the region $R$ bounded by the curve $y = -(x - 1)(x - 3)$ and the $x$-axis.

4.2 A solid $S$ is obtained by rotating $R$ about the $y$-axis. Sketch $S$.

4.3 Consider a vertical strip under the curve at some point $1 \leq x_i \leq 3$ of width $\Delta x$, which is rotated about the $y$-axis to form a shell. Sketch the shell.

4.4 Imagine the shell being cut vertically and opened out flat. Find the volume of the shell.

4.5 Using a Riemann sum and a limit, express the volume of $S$ as an integral. Hence find the volume of $S$. 

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